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# Pairings and monomorphisms of classifying spaces

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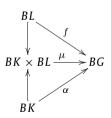
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### ABSTRACT

We consider the maps between classifying spaces of compact Lie groups of the form  $BK \times BL \rightarrow BG$ . If the restriction map  $BL \rightarrow BG$  is a weak epimorphism, then the restriction on BK is known to factor through the classifying spaces of the center of the compact Lie group G. Suppose H is a semi-simple subgroup of a connected compact Lie group G with rank $(H) = \operatorname{rank}(G)$ . Replacing the weak epimorphism  $BL \rightarrow BG$  by the map  $BH \rightarrow BG$ , analogous results are obtained. We also consider some monomorphisms of classifying spaces of compact Lie groups, such as  $BSO(n) \rightarrow BSU(n)$ . Our proof will make use of admissible maps.

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We consider the pairing problem of classifying spaces of compact Lie groups for monomorphisms. Recall in general that for a map  $f: Y \to Z$ , the set of the homotopy classes of axes, denoted by  $f^{\perp}(X, Z)$ , consists of all homotopy classes of maps  $\alpha: X \to Z$  such that there is a map (called *a pairing*)  $\mu: X \times Y \to Z$  with restrictions (*axes*)  $\mu|_X \simeq \alpha$  and  $\mu|_Y \simeq f$ . In the case of classifying spaces [10], we have the following commutative diagram:



Here we denote  $\alpha \in f^{\perp}(BK, BG)$ . Let f = Bi where  $i: L \to G$  is a monomorphism of Lie groups. We will obtain a sufficient condition for  $(Bi)^{\perp}(BK, BG) = 0$  and calculate some pairings.

The first author has studied the pairing problem of classifying spaces for weak epimorphisms. By Theorem 1 of [10], we see that if the restriction map  $BL \rightarrow BG$  is a weak epimorphism, then the restriction on BK factors through the classifying spaces of the center of the compact Lie group *G*. A generalization for *p*-compact groups can be found in [11]. A *p*-compact

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group [6] and [17], is a loop space *X* such that *X* is  $\mathbb{F}_p$ -finite and that its classifying space *BX* is  $\mathbb{F}_p$ -complete. The *p*-completion of a compact Lie group *G* is a *p*-compact group if  $\pi_0 G$  is a *p*-group. It is worth to recall here a group theoretical analog. Suppose  $\rho: L \to G$  and  $\alpha: K \to G$  are homomorphisms. If there is a *pairing* homomorphism  $\mu: K \times L \to G$  with  $\mu|_K = \alpha$  and  $\mu|_L = \rho$ , then the image  $\alpha(K)$  must be contained in the centralizer of  $\rho(L)$  in *G*, denoted by  $C_G(\rho)$ .

**Theorem 1.** Suppose that K is a compact Lie group, and that a connected compact Lie group H is a semi-simple subgroup of a connected compact Lie group G with rank(H) = rank(G). Let  $i : H \hookrightarrow G$  be the inclusion. If  $\alpha \in (Bi)^{\perp}(BK, BG)$ , then the following hold:

- (1) The map  $\alpha$  : BK  $\rightarrow$  BG factors through  $B\pi_0 K$  up to homotopy under the map induced by the projection q :  $K \rightarrow \pi_0 K$ . In particular, if K is connected, the map  $\alpha$  is null homotopic.
- (2) There is a homomorphism  $\rho: \pi_0 K \to G$  such that  $\alpha \simeq B\rho \circ Bq$ , and the image of the homomorphism  $\rho(\pi_0 K)$  is contained in the centralizer  $C_G(H)$ .

**Theorem 2.** For the inclusions  $i: SU(m) \hookrightarrow SU(n)$  and  $j: Sp(m) \hookrightarrow Sp(n)$  with  $m \leq n$ , we have the following:

- (1)  $(Bi)^{\perp}(BSU(k), BSU(n)) = [BSU(k), BSU(n-m)].$
- (2)  $(Bj)^{\perp}(BSp(k), BSp(n)) = [BSp(k), BSp(n-m)].$

A result of [12] shows that for the inclusion  $i: SU(n) \to U(n)$ , if a connected compact Lie group K is semi-simple, then  $(Bi)^{\perp}(BK, BU(n)) = 0$ . Similar results for other classical Lie groups are obtained in [15].

**Theorem 3.** For the inclusion  $i: SO(n) \rightarrow SU(n)$  with  $n \ge 3$ , if G is a connected compact Lie group, then any map in  $(Bi)^{\perp}(BG, BSU(n))$  is null homotopic:

 $(Bi)^{\perp}(BG, BSU(n)) = 0.$ 

We will prove the *p*-completed version of this result. So Theorem 3 is its easy consequence. Some of the results in this paper first appeared in the third author's master thesis.

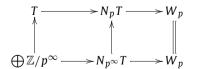
#### 1. Ranks of pairing maps

We will prove Theorem 1 in this section. To do so, we need a few basic results, mostly something about ranks of admissible maps.

**Lemma 1.1.** Let G be semi-simple. For any infinite order element  $\alpha$  of a maximal torus  $T_G$ , there is an element  $\lambda$  contained in the normalizer  $NT_G$  such that  $\lambda \alpha \lambda^{-1} \alpha^{-1}$  is not contained in the center Z(G).

**Proof.** Since *G* is semi-simple, the order of *Z*(*G*) is finite. Let |Z(G)| = n. Now suppose  $\lambda \alpha \lambda^{-1} \alpha^{-1} \in Z(G)$  for any  $\lambda \in NT_G$ . Then we would see  $\lambda \alpha \lambda^{-1} \alpha^{-1} = \zeta$  for some  $\zeta \in Z(G)$ , and hence  $\lambda \alpha^n \lambda^{-1} = (\lambda \alpha \lambda^{-1})^n = \zeta^n \alpha^n = \alpha^n$ . This means that  $\alpha^n$  is fixed by the action of the Weyl group  $W(G) = NT_G/T_G$ . Consequently the infinite set  $\{(\alpha^n)^k | k \in \mathbb{Z}\}$  could be contained in the set  $T_G^{W(G)}$ . This is, however, a finite set, since  $T_G^{W(G)}/Z(G)$  is an elementary abelian 2-group [7, Remark 1.5]. This contradiction completes the proof.  $\Box$ 

Here we first recall the *kernel* of a map  $f : BL \to (BG)_p^{\wedge}$  [9,5], and [16], where  $X_p^{\wedge}$  denotes the *p*-completion of a space *X*. Let *T* (or  $T_L$ ) be a maximal torus of the Lie group *L*. Suppose  $N_pT$  denotes the inverse image in the normalizer of a maximal torus *T* of a *p*-Sylow subgroup  $W_p$  of the Weyl group of *L*. We define a subgroup  $N_p^{\infty}T$  of  $N_pT$  as follows:



Here  $\bigoplus \mathbb{Z}/p^{\infty} \subset T$  is the subgroup of elements whose order is a power of p. The kernel of a map  $f: BL \to (BG)_p^{\wedge}$  is defined in [9, §1] as follows:

$$\operatorname{Ker} f = \{ x \in N_{p^{\infty}}T \mid f|_{B(x)} \simeq 0 \}.$$

Here  $\langle x \rangle$  denotes the subgroup of  $N_{p^{\infty}}T$  generated by *x*. We note that Ker *f* is a group.

Next we recall subgroups related to the center of connected compact Lie groups, [9]. Let Z(L) denote the center of L and let W(L) denote its Weyl group. If L is a simply-connected simple Lie group other than the exceptional Lie group  $G_2$ , we define

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