Topology and its Applications 160 (2013) 264–272

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Pairings and monomorphisms of classifying spaces

Kenshi Ishiguro [a,*], Shotaro Kudo [a], Tomohiro Nakano [b]

[a] Dept. of Applied Mathematics, Fukuoka University, Fukuoka 814-0180, Japan
[b] Wajiro junior high school, Higashi-ku, Fukuoka 814-0201, Japan

ARTICLE INFO

Article history:
Received 22 May 2012
Received in revised form 18 October 2012
Accepted 18 October 2012

MSC:
55R37
55R35
55P60

Keywords:
Classifying space
Lie group
p-Compact group
p-Completion
K-theory
Admissible map

ABSTRACT

We consider the maps between classifying spaces of compact Lie groups of the form $BK \times BL \to BG$. If the restriction map $BL \to BG$ is a weak epimorphism, then the restriction on BK is known to factor through the classifying spaces of the center of the compact Lie group G. Suppose H is a semi-simple subgroup of a connected compact Lie group G with $\mathrm{rank}(H) = \mathrm{rank}(G)$. Replacing the weak epimorphism $BL \to BG$ by the map $BH \to BG$, analogous results are obtained. We also consider some monomorphisms of classifying spaces of compact Lie groups, such as $BSO(n) \to BSU(n)$. Our proof will make use of admissible maps.

We consider the pairing problem of classifying spaces of compact Lie groups for monomorphisms. Recall in general that for a map $f : Y \to Z$, *the set of the homotopy classes of axes*, denoted by $f^{\perp}(X, Z)$, consists of all homotopy classes of maps $\alpha : X \to Z$ such that there is a map (called *a pairing*) $\mu : X \times Y \to Z$ with restrictions (*axes*) $\mu|_X \simeq \alpha$ and $\mu|_Y \simeq f$. In the case of classifying spaces [10], we have the following commutative diagram:

$$\begin{array}{ccc} BL & & \\ \downarrow & \searrow^{f} & \\ BK \times BL & \xrightarrow{\mu} & BG \\ \uparrow & \nearrow_{\alpha} & \\ BK & & \end{array}$$

Here we denote $\alpha \in f^{\perp}(BK, BG)$. Let $f = Bi$ where $i : L \to G$ is a monomorphism of Lie groups. We will obtain a sufficient condition for $(Bi)^{\perp}(BK, BG) = 0$ and calculate some pairings.

The first author has studied the pairing problem of classifying spaces for weak epimorphisms. By Theorem 1 of [10], we see that if the restriction map $BL \to BG$ is a weak epimorphism, then the restriction on BK factors through the classifying spaces of the center of the compact Lie group G. A generalization for p-compact groups can be found in [11]. A p-compact

* Corresponding author.
E-mail addresses: kenshi@cis.fukuoka-u.ac.jp (K. Ishiguro), sd100501@cis.fukuoka-u.ac.jp (S. Kudo).

0166-8641/$ – see front matter
http://dx.doi.org/10.1016/j.topol.2012.10.018

group [6] and [17], is a loop space X such that X is $\mathbb{F}_p$-finite and that its classifying space BX is $\mathbb{F}_p$-complete. The p-completion of a compact Lie group G is a p-compact group if $\pi_0 G$ is a p-group. It is worth to recall here a group theoretical analog. Suppose $\rho : L \to G$ and $\alpha : K \to G$ are homomorphisms. If there is a *pairing* homomorphism $\mu : K \times L \to G$ with $\mu|_K = \alpha$ and $\mu|_L = \rho$, then the image $\alpha(K)$ must be contained in the centralizer of $\rho(L)$ in G, denoted by $C_G(\rho)$.

Theorem 1. *Suppose that K is a compact Lie group, and that a connected compact Lie group H is a semi-simple subgroup of a connected compact Lie group G with $\mathrm{rank}(H) = \mathrm{rank}(G)$. Let $i : H \hookrightarrow G$ be the inclusion. If $\alpha \in (Bi)^{\perp}(BK, BG)$, then the following hold*:

(1) *The map $\alpha : BK \to BG$ factors through $B\pi_0 K$ up to homotopy under the map induced by the projection $q : K \to \pi_0 K$. In particular, if K is connected, the map α is null homotopic.*
(2) *There is a homomorphism $\rho : \pi_0 K \to G$ such that $\alpha \simeq B\rho \circ Bq$, and the image of the homomorphism $\rho(\pi_0 K)$ is contained in the centralizer $C_G(H)$.*

Theorem 2. *For the inclusions $i : SU(m) \hookrightarrow SU(n)$ and $j : Sp(m) \hookrightarrow Sp(n)$ with $m \leqslant n$, we have the following*:

(1) $(Bi)^{\perp}(BSU(k), BSU(n)) = [BSU(k), BSU(n-m)]$.
(2) $(Bj)^{\perp}(BSp(k), BSp(n)) = [BSp(k), BSp(n-m)]$.

A result of [12] shows that for the inclusion $i : SU(n) \to U(n)$, if a connected compact Lie group K is semi-simple, then $(Bi)^{\perp}(BK, BU(n)) = 0$. Similar results for other classical Lie groups are obtained in [15].

Theorem 3. *For the inclusion $i : SO(n) \to SU(n)$ with $n \geqslant 3$, if G is a connected compact Lie group, then any map in $(Bi)^{\perp}(BG, BSU(n))$ is null homotopic*:

$$(Bi)^{\perp}\big(BG, BSU(n)\big) = 0.$$

We will prove the p-completed version of this result. So Theorem 3 is its easy consequence.
Some of the results in this paper first appeared in the third author's master thesis.

1. Ranks of pairing maps

We will prove Theorem 1 in this section. To do so, we need a few basic results, mostly something about ranks of admissible maps.

Lemma 1.1. *Let G be semi-simple. For any infinite order element α of a maximal torus T_G, there is an element λ contained in the normalizer NT_G such that $\lambda\alpha\lambda^{-1}\alpha^{-1}$ is not contained in the center $Z(G)$.*

Proof. Since G is semi-simple, the order of $Z(G)$ is finite. Let $|Z(G)| = n$. Now suppose $\lambda\alpha\lambda^{-1}\alpha^{-1} \in Z(G)$ for any $\lambda \in NT_G$. Then we would see $\lambda\alpha\lambda^{-1}\alpha^{-1} = \zeta$ for some $\zeta \in Z(G)$, and hence $\lambda\alpha^n\lambda^{-1} = (\lambda\alpha\lambda^{-1})^n = \zeta^n\alpha^n = \alpha^n$. This means that α^n is fixed by the action of the Weyl group $W(G) = NT_G/T_G$. Consequently the infinite set $\{(\alpha^n)^k \mid k \in \mathbb{Z}\}$ could be contained in the set $T_G^{W(G)}$. This is, however, a finite set, since $T_G^{W(G)}/Z(G)$ is an elementary abelian 2-group [7, Remark 1.5]. This contradiction completes the proof. $\square$

Here we first recall the *kernel* of a map $f : BL \to (BG)^{\wedge}_p$ [9,5], and [16], where $X^{\wedge}_p$ denotes the p-completion of a space X. Let T (or T_L) be a maximal torus of the Lie group L. Suppose $N_p T$ denotes the inverse image in the normalizer of a maximal torus T of a p-Sylow subgroup W_p of the Weyl group of L. We define a subgroup $N_{p^\infty} T$ of $N_p T$ as follows:

$$\begin{array}{ccccc} T & \longrightarrow & N_p T & \longrightarrow & W_p \\ \uparrow & & \uparrow & & \| \\ \bigoplus \mathbb{Z}/p^\infty & \longrightarrow & N_{p^\infty} T & \longrightarrow & W_p \end{array}$$

Here $\bigoplus \mathbb{Z}/p^\infty \subset T$ is the subgroup of elements whose order is a power of p. The kernel of a map $f : BL \to (BG)^{\wedge}_p$ is defined in [9, §1] as follows:

$$\mathrm{Ker}\, f = \{x \in N_{p^\infty} T \mid f|_{B\langle x\rangle} \simeq 0\}.$$

Here $\langle x\rangle$ denotes the subgroup of $N_{p^\infty} T$ generated by x. We note that $\mathrm{Ker}\, f$ is a group.
Next we recall subgroups related to the center of connected compact Lie groups, [9]. Let $Z(L)$ denote the center of L and let $W(L)$ denote its Weyl group. If L is a simply-connected simple Lie group other than the exceptional Lie group G_2, we define