Topology and its Applications 160 (2013) 341–349

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Codimension zero laminations are inverse limits

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ARTICLE INFO

Article history:
Received 27 April 2012
Accepted 14 November 2012

MSC:
57R30
37B05
37B45

Keywords:
Lamination
Branched manifold
Inverse limit
Equicontinuity

ABSTRACT

The aim of the paper is to investigate the relation between inverse limit of branched manifolds and codimension zero laminations. We give necessary and sufficient conditions for such an inverse limit to be a lamination. We also show that codimension zero laminations are inverse limits of branched manifolds.

The inverse limit structure allows us to show that equicontinuous codimension zero laminations preserves a distance function on transversals.

1. Introduction

Consider the circle $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and the cover of degree 2 of it $p_2(z) = z^2$. Define the inverse limit

$$\mathbf{S}_2 = \varprojlim(\mathbb{S}^1, p_2) = \left\{ (z_k) \in \prod_{k \geqslant 0} \mathbb{S}^1 \;\middle|\; z_k^2 = z_{k-1} \right\}.$$

This space has a natural foliated structure given by the flow $\Phi_t(z_k) = (e^{2\pi i t/2^k} z_k)$. The set $X = \{(z_k) \in \mathbf{S}_2 \mid z_0 = 1\}$ is a complete transversal for the flow homeomorphic to the Cantor set. This space is called *solenoid*. This construction can be generalized replacing $\mathbb{S}^1$ and p_2 by a sequence of compact n-manifolds and submersions between them. The spaces obtained this way are compact laminations with 0-dimensional transversals.

This construction appears naturally in the study of dynamical systems. In [17,18] R.F. Williams proves that an expanding attractor of a diffeomorphism of a manifold is homeomorphic to the inverse limit

$$S \xleftarrow{f} S \xleftarrow{f} S \cdots$$

where f is a surjective immersion of a branched manifold S on itself. A branched manifold is, roughly speaking, a CW-complex with tangent space at each point.

After their introduction by R.F. Williams, branched manifolds and their limits have been extensively used in the study of dynamical systems and foliations. For example W. Thurston uses *train tracks* (1-dimensional branched manifolds) in geodesic

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0166-8641/$ – see front matter
http://dx.doi.org/10.1016/j.topol.2012.11.010

laminations on hyperbolic surfaces [15]. Later, J. Anderson and I. Putnam show [2] that substitution tiling spaces are inverse limits of a CW-complex as in the case of R.F. Williams. J. Bellisard, R. Benedetti and J.-M. Gambaudo [3], F. Gähler [9] and L. Sadun [14] independently extended this result to any tiling, showing that they are inverse limits of branched manifolds. In this case, the projective system has different branched surfaces at each level. With a similar scheme as in [3] R. Benedetti and J.-M. Gambaudo has extended in [4] the previous result to $\mathbb{G}$-solenoids (free actions of a Lie group $\mathbb{G}$ with transverse Cantor structure).

F. Alcalde Cuesta, M. Macho Stadler and the author prove in [1] that any compact without holonomy minimal lamination of codimension zero is an inverse limit, generalizing all previous results.

In this article, we thoroughly explore the relation between inverse limits of branched manifolds and laminations with 0-dimensional transverse structure. Let us start considering the following example. Take the eight figure

$$K = \mathbb{S}^1 \wedge_1 \mathbb{S}^1 = \frac{\mathbb{S}^1 \times \{1,2\}}{(1,1) \sim (1,2)},$$

that is, the two copies of the circle glued by the 1. For each copy of $\mathbb{S}^1$ we have the degree two covering p_2, so we can define $P_2([z,i]) = [z^2, i]$, where $[z,i] \in K$. Let X be the inverse limit $\varprojlim(K, P_2)$. It is easy to see that X is homeomorphic to $\mathbf{S}_2 \wedge_{(1)} \mathbf{S}_2$, i.e., two copies of the solenoid glued by the sequence $(1) \in \mathbf{S}_2$. It is clear that it is not a lamination. The problem is that P_2 does not *iron out* the branching in each step, collapsing the branches at branching points to one single disk. The kind of maps doing that are called *flattening* [3].

We have three main theorems in the paper. Firstly, we show that this is a necessary and sufficient condition on an inverse systems to obtain a lamination as it limit:

Theorem 4.3. *Fix a projective system (B_k, f_k) where B_k are branched n-manifolds and f_k cellular maps, both of class C^r. The inverse limit B_∞ of the system is a codimension zero lamination of dimension n and class C^r if and only if the systems is flattening.*

Secondly, thinking in laminations as tiling spaces [1] we can adapt the constructions for tilings [9,14] to obtain a result in the other direction: from laminations to systems of branched manifolds. This theorem extends [1] to any lamination of codimension zero:

Theorem 5.8. *Any codimension zero lamination $(M, \mathcal{L})$ is homeomorphic to an inverse limit $\varprojlim(S_k, f_k)$ of branched manifolds S_k and cellular maps $f_k : S_k \to S_{k-1}$.*

Finally, the inverse limit structure can give information of the dynamics of the space. M.C. McCord [13] and recently A. Clark and S. Hurder [7] study an important class of solenoidal spaces, those given by real manifolds and regular covering maps as bounding maps. With this structure we can conclude that:

Theorem 6.3. *An equicontinuous lamination of codimension zero preserves a transverse metric.*

2. Branched manifolds

Let $\mathbb{D}^n$ denote the closed n-dimensional unit disk. A *sector* is the, eventually empty, interior of the intersection of a (finite) family of half-spaces through the origin. Fix a finite family of sectors $\mathcal{S}$, and a finite directed tree T with a map $s : VT \to \mathcal{S}$ where VT is the vertex set of T.

Now define a *local branched model* U_T as the quotient set of $\mathbb{D}^n \times VT$ by the relation generated by

$$(x, v) \sim (x, v') \iff \text{the edge } v \to v' \text{ exists in } T \text{ and } x \in \mathbb{D}^n \setminus s(v).$$

The quotient $D_v \subseteq U_T$ of each set $\mathbb{D}^n \times \{v\}$ is called a *smooth disks*. There is a natural map $\Pi_T : U_T \to \mathbb{D}^n$ given by the quotient of the first coordinate projection $pr_1 : (x, v) \mapsto x$, which is a homeomorphism restricted to each smooth disk.

Definition 2.1. A *branched manifold of class C^r of dimension n* is a Polish space S endowed with an atlas of closed disks $\{U_i\}$ homeomorphic to local branched models of dimension n such that there is a cocycle of C^r-diffeomorphisms $\{\alpha_{ij}\}$ between open sets of $\mathbb{D}^n$ fulfilling $\Pi_i \circ \alpha_{ij} = \Pi_j$, where Π_i denotes the natural map of the local models.

Remark 2.2. Definition 2.1 is not the classical one [18]. In our setting the branching behavior is quite simple as we have locally finite branching. This is not true with the classical definition.

Following [18], there is a natural notion of differentiable map: a map $f : S \to S'$ between two branched manifolds of class at least $r \geqslant 1$ is of class C^r if the local map