Topology and its Applications 160 (2013) 382–386

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Topological mixing for cosine operator functions generated by shifts

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ARTICLE INFO

Article history:
Received 25 October 2012
Received in revised form 22 November 2012
Accepted 25 November 2012

MSC:
54H20
47A16
47D09
47B37

Keywords:
Topologically mixing operator
Topologically transitive operator
Hypercyclic operator
ℓ^p-Space

ABSTRACT

Let $1 \leqslant p < \infty$. We characterize topologically mixing cosine operator functions, generated by unilateral and bilateral weighted shifts on $\ell^p(\mathbb{N}_0)$ and $\ell^p(\mathbb{Z})$ respectively. We also give sufficient conditions for such cosine operator functions to be topologically transitive.

1. Introduction

A sequence of bounded linear operators $(T_n)_{n\in\mathbb{N}_0}$ on a Banach space X is called *topologically transitive* if given nonempty open sets $U, V \subset X$, we have $T_n(U) \cap V \neq \emptyset$ for some $n \in \mathbb{N}$. If (T_n) satisfies $T_n(U) \cap V \neq \emptyset$ from some n onwards, then (T_n) is *topologically mixing*. We note that (T_n) is said to be *hypercyclic* if there exists an element $x \in X$ such that the orbit $\{T_n x\}_{n\geqslant 0}$ is dense in X. It has been shown in [10] that (T_n) is topologically transitive if, and only if, it is hypercyclic and has a dense set of hypercyclic vectors. If (T_n) is generated by a single operator T by its iterates, that is $T_n := T^n$, then hypercyclicity is equivalent to transitivity. Transitivity and hypercyclicity for a single operator T and a sequence of operators (T_n) have been studied by many authors; we refer to [2,10,11] for surveys.

In the study of hypercyclicity and transitivity, the shift operators on $\ell^p(\mathbb{N}_0)$ or $\ell^p(\mathbb{Z})$ play an important role for researchers to contract and demonstrate the theory of dynamic of linear operators. For unilateral shifts T on $\ell^p(\mathbb{N}_0)$, Rolewicz [13] in 1969 showed that λT is transitive whenever $|\lambda| > 1$. In [14], Salas successfully gave the sufficient and necessary condition for bilateral weighted shifts on $\ell^p(\mathbb{Z})$ to be transitive. Costakis and Sambarino in [8] characterized mixing bilateral weighted shifts on $\ell^p(\mathbb{Z})$. Feldman [9] simplified Salas' characterization for invertible bilateral weighted shifts. León-Saavedra gave the descriptions for unilateral and bilateral weighted shifts on $\ell^p(\mathbb{N}_0)$ and $\ell^p(\mathbb{Z})$ to be Cesàro hypercyclic in [12]. Common hypercyclic backward shifts on $\ell^p(\mathbb{N}_0)$ were studied in [1]. The characterizations for disjoint hypercyclic weighted shifts on $\ell^p(\mathbb{N}_0)$ and $\ell^p(\mathbb{Z})$ were obtained by Bès and Peris in [3]. Recently [6,7] we extended Salas' results to weighted translations on the Lebesgue space of locally compact groups, which provided a class of transitive and

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[1] The author is supported by NSC of Taiwan under Grant No. NSC 100-2115-M-142-003-MY2.

0166-8641/$ – see front matter
http://dx.doi.org/10.1016/j.topol.2012.11.018

mixing operators on Banach spaces. In this note, we will consider cosine operator functions, generated by unilateral and bilateral weighted shifts, and give sufficient and necessary conditions for such cosine operator functions to be topologically mixing.

Throughout, let $1 \leqslant p < \infty$ and let $\{e_j\colon\ j \in \mathbb{Z}\}$ be the canonical basis of $\ell^p(\mathbb{Z})$. Let $(w_j)_{j\in\mathbb{Z}}$ be a bounded sequence of positive numbers. Then a bilateral weighted forward shift $T : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ is defined by $Te_j = w_j e_{j+1}$ for all $j \in \mathbb{Z}$. If (w_j) is also bounded away from 0, then T is invertible and its inverse $T^{-1} := S : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ is given by $Se_j = \frac{1}{w_{j-1}} e_{j-1}$. Now define a sequence of bounded operators (C_n) by

$$C_n = \frac{1}{2}\big(T^n + S^n\big)$$

for all $n \in \mathbb{Z}$ in which $T^{-n} := (T^{-1})^n = S^n$. Then (C_n) can be regarded as a cosine operator function.

A *cosine operator function* on a Banach space X is a mapping $\mathcal{C}$ from the real line into the space of continuous operators on X satisfying $\mathcal{C}(0) = I$, and the d'Alembert functional equation $2\mathcal{C}(t)\mathcal{C}(s) = \mathcal{C}(t+s) + \mathcal{C}(t-s)$ for all $s, t \in \mathbb{R}$, which implies $\mathcal{C}(t) = \mathcal{C}(-t)$ for all $t \in \mathbb{R}$. Cosine operator functions are crucial for the investigation of semigroups, and we refer to [4,15] for some references. The motivation for the study of (C_n) is inspired by the work in [5]. In [5], Bonilla and Miana obtained a sufficient condition for a cosine operator function $\mathcal{C}(t)$, defined by

$$\mathcal{C}(t) = \frac{1}{2}\big(T(t) + T(-t)\big),$$

to be transitive where T is a strongly continuous translation group on some weighted Lebesgue space $L^p(\mathbb{R})$.

Here we define a cosine operator function $\mathcal{C}$ from $\mathbb{Z}$ into the space of continuous operators by letting $\mathcal{C}(n) = C_n$. Since $C_n = C_{-n}$ for all $n \in \mathbb{Z}$, we will consider the sequence of operators $(C_n)_{n\in\mathbb{N}_0}$, and characterize transitive and mixing cosine operator functions $(C_n)_{n\in\mathbb{N}_0}$.

2. Topological mixing

First, we give the sufficient and necessary condition for such cosine operator functions (C_n) to be topologically mixing.

Theorem 2.1. *Let (w_j) be a bounded sequence of positive numbers, and let T be an invertible bilateral weighted forward shift on $\ell^p(\mathbb{Z})$. Let $C_n = \frac{1}{2}(T^n + S^n)$ where $S = T^{-1}$. Then the following conditions are equivalent.*

(i) *$(C_n)_{n\in\mathbb{N}_0}$ is topologically mixing.*
(ii) *Given $\varepsilon > 0$ and $q \in \mathbb{N}$, there exists some $N \in \mathbb{N}$ such that for all $|j| \leqslant q$ and $n \geqslant N$, we have*

$$\prod_{s=0}^{n-1} w_{j+s} < \varepsilon \quad \textit{and} \quad \frac{1}{\prod_{s=1}^{n} w_{j-s}} < \varepsilon.$$

Proof. (i) $\Rightarrow$ (ii). Let $\varepsilon > 0$. Choose $0 < \delta < \frac{\varepsilon}{2+\varepsilon}$. By the assumption of topological mixing (C_n), there exists some $N > 2q$ such that for each $n \geqslant N$, there is a vector $f \in \ell^p(\mathbb{Z})$ satisfying

$$\Big\| f - \sum_{|j|\leqslant q} e_j \Big\|_p < \delta \quad \text{and} \quad \Big\| C_n f + \sum_{|j|\leqslant q} e_j \Big\|_p < \delta.$$

Without loss of generality, we may assume that f is real-valued by the continuity of the mapping $h \in \ell^p(\mathbb{Z}, \mathbb{C}) \mapsto Re\, h \in \ell^p(\mathbb{Z}, \mathbb{R})$ and the fact T commutes with it. Also, the mapping $h \in \ell^p(\mathbb{Z}, \mathbb{R}) \mapsto h^+ \in \ell^p(\mathbb{Z}, \mathbb{R})$ commutes with T where $h^+ = \max\{0, h\}$. Then

$$\begin{aligned}
\big\|C_n\big(f^+\big)\big\|_p = \big\|(C_n f)^+\big\|_p &= \Big\| \Big(C_n f - \Big(-\sum_{|j|\leqslant q} e_j \Big) + \Big(-\sum_{|j|\leqslant q} e_j \Big) \Big)^+ \Big\|_p \\
&\leqslant \Big\| \Big(C_n f - \Big(-\sum_{|j|\leqslant q} e_j \Big) \Big)^+ \Big\|_p + \Big\| \Big(-\sum_{|j|\leqslant q} e_j \Big)^+ \Big\|_p \\
&= \Big\| \Big(C_n f - \Big(-\sum_{|j|\leqslant q} e_j \Big) \Big)^+ \Big\|_p \leqslant \Big\| C_n f + \sum_{|j|\leqslant q} e_j \Big\|_p < \delta.
\end{aligned}$$

Let $f = \sum_{j\in\mathbb{Z}} \alpha_j e_j$. Then we have

$$\alpha_j > 1 - \delta \quad \big(|j| \leqslant q\big).$$