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On knots and links in lens spaces ☆

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ABSTRACT

In this paper we study some aspects of knots and links in lens spaces. Namely, if we consider lens spaces as quotient of the unit ball B^3 with suitable identification of boundary points, then we can project the links on the equatorial disk of B^3, obtaining a regular diagram for them. In this context, we obtain a complete finite set of Reidemeister type moves establishing equivalence, up to ambient isotopy, a Wirtinger type presentation for the fundamental group of the complement of the link and a diagrammatic method giving the first homology group. We also compute Alexander polynomial and twisted Alexander polynomials of this class of links, showing their correlation with Reidemeister torsion.

1. Introduction

Knot theory is a widespread branch of geometric topology, with many applications to theoretical physics, chemistry and biology. The mainstream of this research has been concentrated for more than one century in the study of knots/links in the 3-sphere, which is the simplest closed 3-manifold, and where the theory is completely equivalent to the one in the familiar space R^3. That study was mainly conducted by the use of regular diagrams, which are suitable projection of the knot/link in a disk/plane. In this way the 3-dimensional equivalence problem is translated into a 2-dimensional equivalence problem of diagrams. Reidemeister proved that two knots/links are equivalent if and only if any of their diagrams can be connected by a finite sequence of three local moves, called Reidemeister moves. Diagrams also help to obtain invariants as the fundamental group of the exterior of the link (also called group of the link), via Wirtinger theorem, while the homology groups, as well as higher homotopy groups, are not relevant in the theory. From the fundamental group other important invariants such as Alexander polynomials (classical and twisted) have been obtained, while from the diagram state sum type invariants have been derived, such as Jones polynomials and quandle invariants.

In the last two decades, studies on knots/links have been generalized in more complicated spaces such as solid torus (see [1,8,9]), or lens spaces, which are the simplest closed 3-manifolds different from the 3-sphere. Particularly important are the class of $(1,1)$-knots (knots in either $\mathbf{S}^3$ called genus one 1-bridge knots) intensively studied by many authors (see [2,3,7,11,16,20]).

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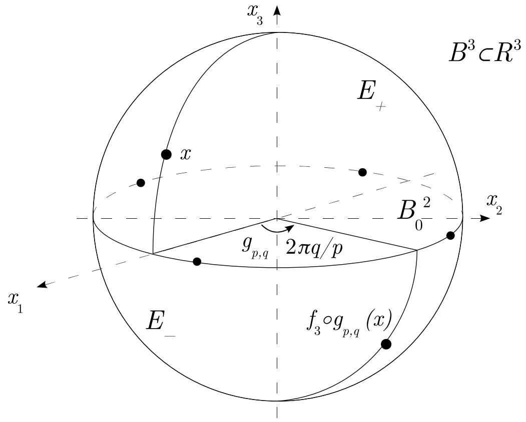

Fig. 1. Representation of $L(p,q)$.

In 1991, Drobotukhina introduced diagrams and moves for knots and links in the projective space, which is a special case of lens space, obtaining in this way an approach to compute a Jones type invariant for these links (see [6]). More recently, Huynh and Le in [12] obtained a formula for the computation of the twisted Alexander polynomial for links in the projective space.

In this paper we extend some of those results for knots/links in the whole family of lens spaces. Our approach use the model of lens spaces obtained by suitable identification on the boundary of a 3-ball described in Section 2, where a concept of regular projection and relative diagrams for the link is defined. In Section 3 we show that the equivalence between links in lens spaces can be translated into an equivalence between diagrams, via a finite sequence of seven type of moves, generalizing the Reidemeister ones. In Section 4 a Wirtinger type presentation for the group of the link is given. In this context the homology groups are not abelian free groups (as in $\mathbf{S}^3$), since a torsion part appears, and in Section 5 a method to compute that directly from the diagram is given. In Section 6 we deal with the twisted Alexander polynomials of these links, finding different properties and exploiting the connection with the Reidemeister torsion.

2. Diagrams

In this paper we work in the *Diff* category (of smooth manifolds and smooth maps). Every result also holds in the *PL* category, and in the *Top* category if we consider only tame links.

A *link* L in a closed 3-manifold M^3 is a 1-dimensional submanifold $L \subset M^3$. Obviously, L is homeomorphic to ν copies of $\mathbf{S}^1$. When $\nu = 1$ the link is called a *knot*. Two links $L', L'' \subset M^3$ are called *equivalent* if there exists an ambient isotopy $H : M^3 \times [0,1] \to M^3$ such that $h_0 = \mathrm{Id}_{M^3}$ and $h_1(L') = L''$, where $h_t(x) = H(x,t)$.

Consider the unit ball $B^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \leqslant 1\}$ and let E_+ and E_- be respectively the upper and the lower closed hemisphere of ∂B^3. Call B_0^2 the equatorial disk, defined by the intersection of the plane $x_3 = 0$ with B^3, and label with N and S respectively the "north pole" $(0,0,1)$ and the "south pole" $(0,0,-1)$ of B^3.

If p and q are two coprime integers such that $0 \leqslant q < p$, let $g_{p,q} : E_+ \to E_+$ be the rotation of $2\pi q/p$ around the x_3-axis, as in Fig. 1, and $f_3 : E_+ \to E_-$ be the reflection with respect to the plane $x_3 = 0$. The *lens space* $L(p,q)$ is the quotient of B^3 by the equivalence relation on ∂B^3 which identifies $x \in E_+$ with $f_3 \circ g_{p,q}(x) \in E_-$. We denote by $F : B^3 \to L(p,q) = B^3/\!\sim$ the quotient map. Note that on the equator $\partial B_0^2 = E_+ \cap E_-$ each equivalence class contains p points.

It is easy to see that $L(1,0) \cong \mathbf{S}^3$ since $g_{1,0} = \mathrm{Id}_{E_+}$. Furthermore, $L(2,1)$ is $\mathbb{RP}^3$, since the above construction gives the usual model of the projective space where opposite points on the boundary of B^3 are identified.

In the following we improve the definition of diagram for links in lens spaces given by Gonzato [10]. Assume $p > 1$, since $L(1,0) \cong \mathbf{S}^3$ is the classical case. Let L be a link in $L(p,q)$ and consider $L' = F^{-1}(L)$. By moving L via a small isotopy in $L(p,q)$, we can suppose that:

i) L' does not meet the poles N and S of B^3;
ii) $L' \cap \partial B^3$ consists of a finite set of points;
iii) L' is not tangent to ∂B^3;
iv) $L' \cap \partial B_0^2 = \emptyset$.[1]

As a consequence, L' is the disjoint union of closed curves in $\operatorname{int} B^3$ and arcs properly embedded in B^3 (i.e., only the boundary points belong to ∂B^3).

[1] The small isotopy that allows L' to avoid the equator ∂B_0^2 is depicted in Fig. 2.