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Cyclic comodules, the homology of j, and j-homology

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ABSTRACT

We provide a somewhat simpler computation of the homology of the image of j spectrum using general techniques for comodules over a coalgebra. We also compute the homotopy of the spectrum $H\mathbb{F}_p \wedge_j H\mathbb{F}_p$, the spectrum which plays the role of $H\mathbb{F}_p \wedge H\mathbb{F}_p$ in the category of j-modules.

1. Introduction

This paper arose as a warm-up to the more difficult computations at the primes 2 and 3 of connective versions of Behrens' $Q(2)$ spectrum, a spectrum conjecturally describing half of the $K(2)$-local sphere [3]. The natural starting case is to understand the $K(1)$-local story, giving rise to the present discussion of the homology of the connective image of J spectrum. The homology results are by no means new: many authors, including Davis, Knapp, and Angeltveit–Rognes have given descriptions of the results [4,7,1]. The presentation herein contains a careful, easier argument using very general properties of comodules over a coalgebra or over a Hopf algebra, and so can be readily adapted to other situations. We have subsequently used this technique to compute the homology of several Postnikov covers of tmf and $p = 2$ and $p = 3$.

Using a very slight modification of an argument of Quillen, we can show that j is a pullback in E_∞ ring spectra [9]. At an odd prime, we have a pullback square

$$\begin{tikzcd} j \arrow[r] \arrow[d] & \ell \arrow[d, "(1\times\psi^r)\circ\Delta"] \\ \ell \arrow[r, "\Delta"'] & \ell \times_{H\mathbb{Z}_p} \ell, \end{tikzcd}$$

where ℓ is the connective Adams summand of p-adic K theory, where Δ is the "diagonal" map from ℓ to $\ell \times_{H\mathbb{Z}_p} \ell$, the pullback of ℓ and ℓ over their zeroth Postnikov section, and where ψ^r is an appropriately chosen Adams operation. At the prime two, the diagram is similar: we replace $\ell \times_{H\mathbb{Z}_p} \ell$ with $ko \times_{X_2} ko$, where X_2 is the three stage Postnikov tower for ko.

This guarantees that j is an E_∞ ring spectrum, allowing us to consider the category of j-modules and therein do computations. In particular, we present a computation of the homotopy of $H\mathbb{F}_p \wedge_j H\mathbb{F}_p$ for odd primes p, a Hopf algebra

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suitable for computing the homotopy of a j-module using a modified Adams spectral sequence strikingly similar to that employed by the author in [6].

2. Cyclic comodules

In this section, we recast some easy and well-known statements about modules over an algebra A into statements about comodules over a graded coalgebra of finite type C. All algebras and coalgebras are taken over a ground field k, and everything considered will be graded, connected and finite type. In particular, Hom is a functor to graded vector spaces over k. We begin with dual versions of generators and finite generation.

Definition 2.1. Let N be a comodule with coaction ψ. A set of cogenerators for N is a collection $\{f_i \mid i \in I\}$ of elements of N^* such that the map $N \to \prod_I C$ defined by the composite

$$N \xrightarrow{\psi} C \otimes_k N \xrightarrow{1\otimes\prod f_i} \prod_I C$$

is injective.

A comodule is finitely cogenerated if the set I can be chosen to be finite.

A comodule is cyclic if the set I can be chosen to be the one point set.

These definitions are simply the linear dual of the corresponding statements for modules over an algebra. We will focus almost exclusively from this point on cyclic comodules. We begin with a weak form of a change of rings theorem. For notation, if M and N are C-comodules, let $\mathrm{Hom}_C(M,N)$ denote the k vector space of C-comodule maps from M to N. If there is no ambiguity, we will often drop the subscript C.

Lemma 2.2. *Let C and A be coalgebras, and let $f: C \to A$ be a map of coalgebras. If M is a left A-comodule and N a left C-comodule, then there is a natural isomorphism*

$$\mathrm{Hom}_C(N, C \,\square_A\, M) \cong \mathrm{Hom}_A(N, M),$$

where N is given an A-comodule structure via the map f.

Proof. This statement is dual to the statement that if $f: A \to C$ is a map of algebras, then we have a natural isomorphism

$$\mathrm{Hom}_{C\text{-mod}}(C \otimes_A M, N) \cong \mathrm{Hom}_{A\text{-mod}}(M, N).$$

Since for every finite type comodule C and finite type C-comodules N and M we have a natural isomorphism

$$\mathrm{Hom}_{C^*\text{-mod}}\big(M^*, N^*\big) \cong \mathrm{Hom}_C(N, M),$$

we conclude the theorem. $\square$

In particular, if $A = k$ in the previous lemma, then since $C = C \,\square_k\, k$, we have a description of $\mathrm{Hom}_C(N, C)$.

Corollary 2.3. *There is a natural isomorphism*

$$\mathrm{Hom}_C(N, C) \cong N^*$$

given by composition with the augmentation $\epsilon: C \to k$.

Since cyclic comodules can be viewed as subcomodules of C, we can identify $\mathrm{Hom}(N, M)$ with a subspace of N^* whenever M is a cyclic comodule.

This set-up allows us to easily identify large pieces of the kernel of a map.

Lemma 2.4. *Let $F \in \mathrm{Hom}(N, C)$ correspond to a functional $f \in N^*$. If N' is a subcomodule of N that is in the kernel of f, then N' is in the kernel of F.*

Proof. Let ψ be the coaction on N. The map F is defined by

$$F = (1 \otimes f) \circ \psi.$$

Since N' is a subcomodule, $\mathrm{Im}(\psi|_{N'}) \subset C \otimes N'$, and this is annihilated by f. $\square$

We conclude the section by strengthening several assumptions and getting some results about multiplicative structures. Let A be a connected Hopf algebra over k, and let N be a subcomodule of A which is also a subalgebra. All of the examples we will consider subsequently are of this form.