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Interior components of a tile associated to a quadratic canonical number system

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Abstract

Let $\alpha = -2 + \sqrt{-1}$ be a root of the polynomial $p(x) = x^2 + 4x + 5$. It is well known that the pair $(p(x), \{0, 1, 2, 3, 4\})$ forms a *canonical number system*, i.e., that each $x \in \mathbb{Z}[\alpha]$ admits a finite representation of the shape $x = a_0 + a_1\alpha + \cdots + a_\ell\alpha^\ell$ with $a_i \in \{0, 1, 2, 3, 4\}$. The set $\mathcal{T}$ of points with integer part 0 in this number system

$$\mathcal{T} := \left\{\sum_{i=1}^{\infty} a_i\alpha^{-i},\ a_i \in \{0, 1, 2, 3, 4\}\right\}$$

is called the *fundamental domain* of this canonical number system. It has been studied extensively in the literature. Up to now it is known that it is a plane continuum with nonempty interior which induces a tiling of the plane. However, its interior is disconnected. In the present paper we describe some of (the closures of) the components of its interior as attractors of graph directed self-similar constructions. The associated graph can also be used in order to determine the Hausdorff dimension of the boundary of these components. Amazingly, this dimension is strictly smaller than the Hausdorff dimension of the boundary of $\mathcal{T}$.

Keywords: Canonical number systems; Tiles; Systems of graph directed sets

1. Introduction and basic definitions

We are interested in describing the topology of a plane self-similar set with disconnected interior that is related to a quadratic canonical number system (see Fig. 1). More precisely, we want to describe the closure of some components of its interior by a graph directed self-similar set. Moreover, we are able to calculate the Hausdorff dimension of the boundary of these components. In general, it seems to be difficult to study fine topological properties of self-similar

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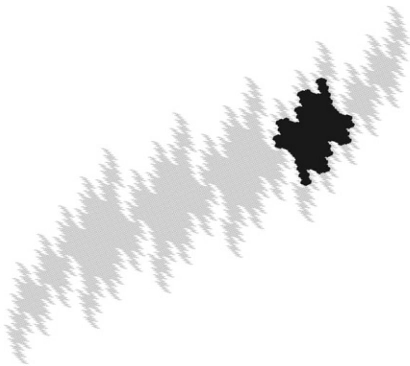

Fig. 1. Tile associated to the base $-2+\sqrt{-1}$ with interior component containing 0.

fractals. This paper aims to present a paradigm[2] and we hope that the methods we used will be capable of being generalized to admit a more systematic study of components of self-similar sets.

We start with the necessary definitions. It is known (see [9,14,15]) that the root $\alpha=-2+\sqrt{-1}$ of the polynomial x^2+4x+5 together with $\mathcal{N}:=\{0,1,2,3,4\}$ forms a *canonical number system* (or *CNS*) $(\alpha,\mathcal{N})$, i.e., each element $x\in\mathbb{Z}[\alpha]$ has a unique representation

$$x=\sum_{i=0}^{\ell(x)} a_i\alpha^i$$

for some non-negative integer $\ell(x)$ and $a_i\in\mathcal{N}$ with $a_{\ell(x)}\neq 0$ for $x\neq 0$. We define the natural embedding

$$\begin{aligned}\Phi:\mathbb{C}&\to\mathbb{R}^2\\ x&\mapsto\big(\Re(x),\Im(x)\big).\end{aligned}$$

Then the multiplication by α can be represented by the 2×2 matrix

$$\mathbf{A}:=\begin{pmatrix}-2 & -1\\ 1 & -2\end{pmatrix},$$

i.e., for every $x\in\mathbb{C}$,

$$\Phi(\alpha x)=\mathbf{A}\Phi(x).$$

The set $\mathcal{T}$ of points of integer part zero in the base α embedded into the plane is defined by

$$\mathcal{T}:=\left\{\sum_{i=1}^{\infty}\Phi\big(\alpha^{-i}a_i\big),\ (a_i)_{i\in\mathbb{N}}\in\mathcal{N}^{\mathbb{N}}\right\}=\left\{\sum_{i=1}^{\infty}\mathbf{A}^{-i}\Phi(a_i),\ (a_i)_{i\in\mathbb{N}}\in\mathcal{N}^{\mathbb{N}}\right\}\tag{1.1}$$

and is depicted in Fig. 1. $\mathcal{T}$ is often called the *fundamental domain* of the number system $(\alpha,\mathcal{N})$. Thus each point of this set can be represented by an *infinite string* $w=(a_1,a_2,a_3,\dots)$ with $a_i\in\mathcal{N}$. The set $\mathcal{T}$ satisfies the equation

$$\mathcal{T}=\bigcup_{i=0}^{4}\psi_i(\mathcal{T}),\tag{1.2}$$

where ψ_i $(i=0,\dots,4)$ are contractions defined via the matrix $\mathbf{A}$ and the embedding Φ by

$$\psi_i(x)=\mathbf{A}^{-1}\big(x+\Phi(i)\big),\quad x\in\mathbb{R}^2\ (0\leqslant i\leqslant 4).\tag{1.3}$$

$\mathcal{T}$ is a self-similar connected compact set (or *continuum*) with nonempty interior (see [13]). It induces a *tiling* of the plane by its translates. We recall that a *tiling* (cf. [11,25]) of the plane is a decomposition of $\mathbb{R}^2$ into sets whose

[2] See also Bailey et al. [2] where the components of the interior of the Lévy dragon are studied; interestingly, their structure is totally different from the ones studied in the present paper.