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Topology vs. algebra: Themes in the research of Mel Henriksen

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ABSTRACT

We discuss themes in our joint research with Mel Henriksen. These include the prime ideal structure of $\mathcal{C}(X)$, regular ring extensions of $\mathcal{C}(X)$, and the construction of various "covers" (generalizations of absolutes) of Tychonoff spaces. We also reflect on Mel Henriksen the man.

1. Introduction

In the words of the old "Reader's Digest" articles, Mel Henriksen was one of the most unforgettable people that I have ever met. His enthusiasm for, and curiosity about, mathematics were unmatched among the mathematicians I have known over a research career spanning forty-five years. He had a genius for pushing his co-authors to work hard and productively. He had a talent for asking the right questions, and his energy level was beyond that of any of my other colleagues. He inspired others.

Mel loved (most) people, but had zero tolerance for what he perceived to be bafflegab, particularly that emanating from journal editors, referees, and university administrators. This resulted in his being involved in numerous battles with all of the above. These battles were not motivated by self-interest, but rather by his uncompromising vision of what good universities and good journals ought to be. He struggled and spoke out where many of us would have sighed and taken the path of least resistance. He did not go gentle into that good night.

Mel worked at the frontier where ring theory meets general topology. He and I co-authored (frequently with others) thirteen published research articles over his career (see [7–19] in the references). I think that Mel was, at heart, more of an algebraist than a topologist, whereas I am a topologist who knows a very modest amount of ring theory. Our complementary strengths made for fruitful collaborations. In what follows I attempt to describe some common themes that run through many of our papers, and to illustrate these themes with examples from our papers.

I thank the Guest Editors for their invitation to write this paper. I apologize to those whose research, related to topics discussed herein, has been noted only briefly or perhaps not at all. Space considerations compel brevity.

2. Topology of X vs. algebra of $\mathcal{C}(X)$

A fundamental question considered in many of Mel's papers is as follows. Let X be a Tychonoff topological space, and denote by $\mathcal{C}(X)$ the ring of continuous real-valued functions with domain X. (In fact $\mathcal{C}(X)$ is a lattice-ordered ring, and sometimes its order structure plays a role in the questions being considered.) How do the topological properties of X

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determine the ring-theoretic properties of $\mathcal{C}(X)$? How do the ring-theoretic properties of $\mathcal{C}(X)$ determine the topological properties of X?

Several papers illustrate the research that such questions generated. In [9] we studied *cozero-complemented* spaces. Recall that a subset of a space X is called a *zero-set* if it is of the form $\{x \in X\colon\ f(x) = 0\}$ (also denoted $Z(f)$) for some $f \in \mathcal{C}(X)$. The complement $X \setminus Z(f)$, also denoted $coz(f)$, of a zero-set is called a *cozero*-set. If V is a cozero-set of X, a cozero-set W of X is called a *cozero-complement* of V if $V \cup W$ is dense in X and $V \cap W = \phi$. The space X is called *cozero-complemented* if each cozero-set of X has a (not necessarily unique) cozero-complement. At Mel's instigation, he and I explored this property: under what circumstances was it inherited by subspaces, preserved by products, or preserved directly and inversely by certain sorts of continuous surjections? The fruits of our investigations appear in [9]. Shortly thereafter Gary Gruenhage [6] extended our work and answered many of the open questions raised in our paper.

Mel did not pluck the concept of cozero-complemented spaces out of the air. His motivation for considering this class of spaces arose because a space X is cozero-complemented iff the space $Min\,\mathcal{C}(X)$, the space of minimal prime ideals of $\mathcal{C}(X)$ endowed with the hull-kernel topology, is compact. In fact, 1.3 of [9] provides a list of characterizations of cozero-complementation gathered from diverse sources; some speak to the topology of X, others to the ring-theoretic properties of $\mathcal{C}(X)$. Mel loved this sort of interplay.

Mel was particularly interested in the prime ideal structure of $\mathcal{C}(X)$. Those of us who worked with him frequently shared this interest, even though we often did not realize it until Mel informed us of this hitherto unknown item on our list of research priorities. Recall that a *P-space* is a space X in which every prime ideal of $\mathcal{C}(X)$ is maximal – equivalently, a space X for which $\mathcal{C}(X)$ is a von Neumann regular ring, or a space X in which each zero-set is open (hence clopen). These were introduced in the 1950s by Mel and Leonard Gillman – see [4] for references and more topological characterizations of P-spaces. Another way of characterizing P-spaces is to say that each chain of prime ideals in $\mathcal{C}(X)$ has length 1. One of several ways to generalize the definition of a P-space is to require that each chain of prime z-ideals of $\mathcal{C}(X)$ be of finite length. In [10], together with Jorge Martinez, we took the obvious first step by studying those spaces X for which each such chain has length at most 2. We call these *quasi-P spaces*. Among our results is this: a locally compact normal space is quasi-P iff X is scattered with Cantor–Bendixon index no greater than 2. We study subspaces and products of quasi-P spaces, cozero-complemented quasi-P spaces, and preservation of the property directly and inversely by certain sorts of continuous maps. Although the defining concept is stated algebraically, most of the research machinery in these investigations is topological. Mel and I studied other generalizations of P-spaces in [7] and [18].

Finally (in this section), I briefly mention the only paper I have written with five co-authors, namely [12]. Its genesis is as follows. In pursuit of a now-forgotten goal, Mel and I wanted to use the "fact" that if X and Y are Tychonoff spaces, then the cardinality $|\mathcal{C}(X \times Y)|$ of $\mathcal{C}(X \times Y)$ is the larger of $|\mathcal{C}(X)|$ and $|\mathcal{C}(Y)|$. This clearly is true if X and Y are compact. We decided that we should verify this "fact". However, building on work by Comfort and Hager [1], we soon found that if D is a discrete space of cardinality t, and if X is a Tychonoff space for which $\mathcal{C}(X)$ has cardinality m, then (see 3.2 of [12])

$$\big|\mathcal{C}(X \times D)\big| > \big|\mathcal{C}(X)\big|\big|\mathcal{C}(D)\big| \quad \Longleftrightarrow \quad m^t > m^\omega = m \geqslant 2^t.$$

With a modest amount of effort, many such pairs of cardinals can be found. Then we developed a machine that allowed us, for example, to produce two first countable Lindelöf spaces X and Y for which

$$\big|\mathcal{C}(X)\big| = \big|\mathcal{C}(Y)\big| = c, \quad \text{while } \big|\mathcal{C}(X \times Y)\big| = 2^c.$$

Others joined in the hunt, and expanded the scope of the project far beyond its modest beginnings. But once again, it was Mel who initiated the project.

3. Build a new ring, build a new space

Another theme in the study of rings of continuous functions is the following. Let F be a functor from $\mathcal{A}$ to $\mathcal{A}$, where $\mathcal{A}$ denotes the category of commutative rings with 1 and ring homomorphisms. If X is a Tychonoff space then $F(\mathcal{C}(X))$ is an object in the category $\mathcal{A}$, *i.e.*, another commutative ring. It is natural (if you were Mel) to ask: given an X, can we find a Tychonoff space Y such that $F(\mathcal{C}(X)) = \mathcal{C}(Y)$? Perhaps there will be such a Y only for some X; can we characterize those X? What is "nice" way to describe, or construct, Y in terms of X?

In [11] R. Raphael, Mel and I studied an example of such a construction. This study grew out of an investigation by Raphael and myself of the epimorphic hull of $\mathcal{C}(X)$; see [22]. Recall that a commutative ring A is (*von Neumann*) *regular* if,

$$\forall a \in A, \exists \hat{a} \in A \quad \text{such that } a^2\hat{a} = a.$$

If we define a^* to be $a\hat{a}^2$, then a^* is the unique element b in A that satisfies both

$$a^2 b = a \quad \text{and} \quad b^2 a = b.$$

As noted above, $\mathcal{C}(X)$ is regular iff X is a P-space. In this case, if $f \in \mathcal{C}(X)$ then f^* is given by

$$f^*(x) = \begin{cases} \frac{1}{f(x)}, & x \in coz\, f; \\ 0, & x \in Z(f). \end{cases}$$