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Between compactness and completeness

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This paper is dedicated to Som Naimpally

Abstract

Call a sequence in a metric space cofinally Cauchy if for each positive ε there exists a cofinal (rather than residual) set of indices whose corresponding terms are ε-close. We give a number of new characterizations of metric spaces for which each cofinally Cauchy sequence has a cluster point. For example, a space has such a metric if and only each continuous function defined on it is uniformly locally bounded. A number of results exploit a measure of local compactness functional that we introduce. We conclude with a short proof of Romaguera's Theorem: a metrizable space admits such a metric if and only if its set of points having a compact neighborhood has compact complement.

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1. Introduction

All mathematicians are familiar with compact metric spaces and complete metric spaces and their standard properties. Between these lies the class of *boundedly compact metric spaces*—spaces in which closed and bounded sets are compact, to which Euclidean spaces belong. One invariably learns the following facts about a compact metric space $\langle X, d\rangle$: (1) each continuous function defined on X with values in an arbitrary metric space $\langle Y, \rho\rangle$ is uniformly continuous; (2) each pair of disjoint closed nonempty subsets of X lie a positive distance apart; and (3) each open cover of X has a Lebesgue number. While none of these properties are characteristic properties of compact spaces, they are each characteristic properties of a larger class of spaces most frequently called UC spaces in the literature. It is obvious that the UC spaces neither contain nor are contained in the spaces in which closed and bounded sets are compact: an infinite set equipped with the zero–one metric belongs to the former but not the latter, whereas n-dimensional Euclidean space in which closed and bounded sets are compact is not UC.

The UC spaces, first systematically studied by Atsuji [2], have been the subject of a number of articles over the years, most recently the survey article [17], where they are called *Atsuji spaces*, following [4,5]. Occasionally, they have been called *normal metric spaces* [19] and *Lebesgue metric spaces* [20,24] in the literature. One sequential characterization that was discovered early on is this: if $\langle x_n\rangle$ is a sequence in X with $\lim d(x_n, \{x_n\}^c) = 0$, then $\langle x_n\rangle$

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has a cluster point [2,15]. This implies immediately that the set of limit points of a UC space is compact. But there is a second sequential characterization of UC spaces discovered by Toader [26] that clarifies their relation to complete spaces as distinct from compact ones. A sequence $\langle x_n \rangle$ is of course Cauchy if $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall \{n, j\} \subseteq \mathbb{N}$, $n > n_0$, and $j > n_0 \Rightarrow d(x_n, x_j) < \varepsilon$. If we permute the inner two quantifiers we call the result a *pseudo-Cauchy sequence* [5]:

$$\forall \varepsilon > 0\ \forall n_0 \in \mathbb{N}\ \exists \{n, j\} \subseteq \mathbb{N} \quad \text{with } n > n_0 \text{ and } j > n_0 \text{ and } d(x_n, x_j) < \varepsilon.$$

A metric space is evidently complete if and only if each Cauchy sequence with distinct terms has a cluster point. Toader proved that $\langle X, d \rangle$ is a UC space if and only if each pseudo-Cauchy sequence with distinct terms has a cluster point.

Toader's pseudo-Cauchy sequences are those for which pairs of terms are arbitrarily close frequently. But there is a second natural way to generalize the definition of Cauchy sequence [13,14]. A sequence is Cauchy if for each $\varepsilon > 0$, there exists a residual set of indices $\mathbb{N}_\varepsilon$ such that each pair of terms whose indices come from $\mathbb{N}_\varepsilon$ are within ε of each other. If we replace *residual* by *cofinal* then we obtain sequences that we here call cofinally Cauchy.

Definition 1.1. A sequence $\langle x_n \rangle$ in a metric space $\langle X, d \rangle$ is called *cofinally Cauchy* if $\forall \varepsilon > 0$ there exists an infinite subset $\mathbb{N}_\varepsilon$ of $\mathbb{N}$ such that for each $n, j \in \mathbb{N}_\varepsilon$ we have $d(x_n, x_j) < \varepsilon$.

It is the purpose of this paper to cast new light on those metric spaces in which each cofinally Cauchy sequence has a cluster point, a collection of spaces that we call here the cofinally complete metric spaces. Our results reveal that such spaces position themselves relative to uniformly locally compact spaces in the same way that UC spaces sit relative to the uniformly discrete spaces, and we exhibit more generally a striking parallelism between the two classes of spaces. Central in our analysis is a measure of local compactness functional that parallels the index of isolation so important in the study of UC spaces and that leads to Cantor-type theorems. We also characterize such spaces in terms of a uniform property that continuous functions defined on them must have, and produce a short proof of Romaguera's Theorem [23, Thm. 2]: a metrizable space has a compatible cofinally complete metric if and only if its set of points having no compact neighborhood is compact.

Cofinal completeness can of course be formulated in terms of nets and entourages and it is in this more general form that it was considered first implicitly by Corson [8] and then by Howes [13] who showed that a completely regular Hausdorff space is paracompact if and only if it admits a compatible cofinally complete uniformity. A few years later, Rice [22] introduced the notion of uniform paracompactness for a Hausdorff uniform space X: for each open cover $\{V_i\colon i \in I\}$ of X there exists an open refinement and an entourage U such that for each $x \in X$, $U(x)$ meets only finitely many members of the refinement. Subsequently, the reviewer of Rice's paper [25] observed that uniform paracompactness is equivalent to net cofinal completeness for a Hausdorff uniform space (see also [14, Thm. 4.6]).

2. Preliminaries

First we list some notational conventions. Let x_0 be a point in a metric space $\langle X, d \rangle$ and let $\varepsilon > 0$. We write $S_\varepsilon(x_0)$ (resp., $S_\varepsilon[x_0]$) for the open (resp., closed) ε-ball with center x_0. If A is a nonempty subset of X, we write $d(x_0, A)$ for the distance from x_0 to A, and if $A = \emptyset$ we agree that $d(x_0, A) = \infty$. We denote the open *ε-enlargement* of A by A^ε, i.e.,

$$A^\varepsilon = \{x\colon d(x, A) < \varepsilon\} = \bigcup_{x \in A} S_\varepsilon(x).$$

If A, B are subsets of X, we define the *Hausdorff distance* [16,6] between them by

$$\begin{aligned} H_d(A, B) &= \max\{\sup\{d(a, B)\colon a \in A\}, \sup\{d(b, A)\colon b \in B\}\} \\ &= \inf\{\varepsilon > 0\colon B \subseteq A^\varepsilon \text{ and } A \subseteq B^\varepsilon\}. \end{aligned}$$

Restricted to the nonempty closed subsets of X, Hausdorff distance so defined is an extended real-valued metric which is finite valued when restricted to the nonempty closed and bounded sets. Of course, $x \to \{x\}$ is an isometry.

If A is a subset of X we write $\operatorname{diam}(A)$, $\operatorname{int}(A)$, $\operatorname{cl}(A)$, $\operatorname{bd}(A)$ and A' for the diameter, interior, closure, boundary and set of limit points of A. Perhaps the most visual characterization of UC spaces involves the set of limit points of X itself [2,15,6]: X' is compact and $\forall \varepsilon > 0\ \exists \delta > 0$ such that $\{x, w\} \cap (X')^\varepsilon = \emptyset \Rightarrow d(x, w) \geqslant \delta$. Atsuji introduced