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The terminal hyperspace of homogeneous continua

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ABSTRACT

We investigate the structure of the collection of terminal subcontinua in homogeneous continua. The main result is a reduction of this structure to six specific types. Three of these types are of one-dimensional spaces, and examples representing these types are known. It is not known whether higher dimensional examples having non-trivial terminal subcontinua and representing the three remaining types exist.

In 1948, R.H. Bing made a major breakthrough [2] in the study of homogeneous continua by showing the homogeneity of Moise' *pseudo-arc* [18]. The pseudo-arc is *hereditarily indecomposable*; in fact it is homeomorphic to earlier defined *Knaster's continuum* [3,10]. By this example subcontinua later called *terminal* were introduced to the study of homogeneous spaces. Though terminal continua in homogeneous spaces may seem counterintuitive, every subcontinuum of a hereditarily indecomposable continuum, such as the pseudo-arc, is terminal. In 1955, F.B. Jones published [8] his aposyndetic decomposition theorem for homogeneous continua. The elements of this decomposition are terminal. The *circle of pseudo-arcs* defined in 1959 by Bing and Jones [4] is a homogeneous continuum having the aposyndetic decomposition non-trivial, and all sufficiently small subcontinua terminal. Terminal subcontinua of homogeneous continua became a subject of extensive study. James T. Rogers showed [24] that homogeneous continua having all subcontinua terminal are tree-like. Conversely, the author showed, in a joint paper with Paweł Krupski [11], that homogeneous tree-like continua are hereditarily indecomposable, that is, they have all subcontinua terminal. A large class of new examples of homogeneous continua having all sufficiently small subcontinua terminal was defined by Wayne Lewis [14]. Rogers [25–27] studied continuous decompositions of homogeneous continua into terminal subcontinua. Terminal subcontinua of homogeneous spaces were also the subject of study by Tadeusz Maćkowiak and Edward Tymchatyn [16], Maćkowiak [15], Charles L. Hagopian [7], Zhou Youcheng and Lin Shou [31], and others. Recently, Rogers [28] has shown that the quotient space of a homogeneous continuum under

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the aposyndetic decomposition with non-trivial fibers has dimension at most 1. This new result is an essential tool in this paper.

The study presented in this paper was initiated in 1999, in collaboration with Carl Seaquist, when the author was a visiting faculty at Texas Tech University in Lubbock. At this early stage we were looking for new properties of the set of terminal subcontinua in homogeneous spaces, and asking questions such as: Is the set of the components of the hyperspace of terminal subcontinua always finite? If not, is it countable? Some of the concepts used in this paper, such as a *locally minimal terminal subcontinuum* and an *intrinsic decomposition*, appeared during these discussions (under different names). This project was abandoned when the author left Texas Tech in 2000. The results presented in this paper are new but the significance of the early discussions and observations made in collaboration with Carl Seaquist should be acknowledged.

In this paper we answer many questions earlier asked in Lubbock. For instance, we show that the terminal hyperspace of a homogeneous continuum can have at most three components. Of these three components only one can have two different members Y_1 and Y_2 such that $Y_1 \subset Y_2$. Moreover, in this last case there is a unique order arc of terminal continua from Y_1 to Y_2. The study presented in this paper can be viewed as a step towards solving the following major open problem (compare related [13, Problems 87, 92 and 93]):

Question 1. If X is a homogeneous continuum and Y its proper terminal subcontinuum, is Y at most one-dimensional?

The main result asserts that the structure of the terminal hyperspace of a homogeneous continuum must represent one of the six types schematically pictured in Fig. 1 in the end of the paper. Each known example has type (a), (b) or (c). If a homogeneous continuum of type (d), (e) or (f) exists, Question 1 will be answered in the negative.

The tools developed in this paper can also be used in the future study of other classes of subcontinua of homogeneous continua such as *semi-terminal* ones [22,23].

1. Preliminaries

In this paper, all spaces are metric and mappings continuous. A space X is called *homogeneous* if for every $x, y \in X$ there is a homeomorphism $h : X \to X$ such that $h(x) = y$. An *ε-map* is a map $f : X \to Y$ between spaces X and Y such that $\operatorname{diam} f^{-1}(y) < \varepsilon$ for each $y \in f(X)$. If X is a space, 2^X denotes the collection of non-empty compact subsets of X, and $C(X)$, the collection of subcontinua of X, both equipped with the Hausdorff metric. For a compact space X, whenever in the paper we refer to a "closed collection of continua" or "closed collection of closed sets" we mean closed subsets of $C(X)$ or 2^X. Since 2^X and $C(X)$ are compact if X is, such collections are always compact. If X is compact, $H(X)$ stands for the group of self-homeomorphisms of X with the *sup* metric.

A one-dimensional continuum is called a *curve*. A *Kelley continuum* [9] is a continuum X such that for every $K \in C(X)$, every $p \in K$ and every sequence $\{p_n\} \subset X$ converging to p, there are continua K_n converging to K in $C(X)$ with $p_n \in K_n$ for each n. Each homogeneous continuum is Kelley [30].

If X is a homogeneous compact space, then for every positive ε there is a number δ, called an *Effros number* for ε, such that for each $x, y \in X$ with $d(x, y) < \delta$, there is some homeomorphism $h \in H(X)$ such that $h(x) = y$ and $d(z, h(z)) < \varepsilon$ for each $z \in X$. This is called the *Effros theorem*. It follows from the more general statement that for each $x \in X$, the evaluation map, $h \mapsto hx$, from the homeomorphism group $H(X)$ onto X is open. The latter follows from [6, Theorem 2]. (See also [29, Theorem 3.1].) By [5, Theorem 3.3], we have a similar result for Borel subgroups of $H(X)$.

2. Basic concepts

If X is a topological space, and $\mathcal{P}$ a collection of non-empty subsets of X, we say that $\mathcal{P}$ is an *intrinsic collection* provided $h(P) \in \mathcal{P}$ for every self-homeomorphism $h : X \to X$ of X and $P \in \mathcal{P}$. If $\mathcal{P}$ is intrinsic in X and partitions X, then other authors often call such $\mathcal{P}$ "a partition (or decomposition) respected by the group of self-homeomorphisms of X." Here we call such $\mathcal{P}$ an *intrinsic partition* or an *intrinsic decomposition* of X. A partition $\mathcal{P}$ of X is called *homogeneous* provided for every $x, y \in X$ there is a homeomorphism $h : X \to X$ such that $h(x) = y$ and $h(P) \in \mathcal{P}$ for each $P \in \mathcal{P}$. In other words, the partition $\mathcal{P}$ is homogeneous if and only if the group of self-homeomorphisms of X that respect $\mathcal{P}$ acts transitively on X. It is easy to see that upper semi-continuous homogeneous decompositions of compact spaces into closed subsets have both the members and quotient spaces homogeneous. Clearly, only homogeneous spaces can have homogeneous partitions. An intrinsic partition of a homogeneous space is always homogeneous. If $\mathbb{S}^1$ is the unit circle, and $X = \mathbb{S}^1 \times \mathbb{S}^1$ the torus, the decomposition of X into the circles $\mathbb{S}^1 \times \{s\}$ is homogeneous but not intrinsic.

Definition 2.1. A set Y in a space X is called *fastened* provided that $h(Y) = Y$ for every homeomorphism $h : X \to X$ such that $h(Y) \cap Y \neq \emptyset$.

We observe the following.

Proposition 2.2. *The members of an intrinsic decomposition of a space X are fastened in X.*