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Generating varieties, Bott periodicity and instantons

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ABSTRACT

Let G be the classical group and let $\mathcal{M}_k(G)$ be the based moduli space of G-instantons on S^4 with instanton number k. It is known that $\mathcal{M}_k(G)$ yields real and symplectic Bott periodicity, however an explicit geometric description of the homotopy equivalence has not been known. We consider certain orbit spaces in $\mathcal{M}_k(G)$ and show that the restriction of the inclusion of $\mathcal{M}_k(G)$ into the moduli space of connections, which, in turn, is explicitly described by the commutator map of G. We prove this restriction satisfies a triple loop space version of the generating variety argument of Bott (1958) [5], and it also gives real and symplectic Bott periodicity. This also gives a new proof of real and symplectic Bott periodicity.

1. Introduction

Let G be a compact connected simple Lie group. Then there is an isomorphism $\pi_3(G) \cong \pi_4(BG) \cong \mathbb{Z}$. We will fix an isomorphism $\pi_3(G) \cong \mathbb{Z}$. Then principal G-bundles over S^4 are classified by $\mathbb{Z} = \pi_3(G)$, and denote by P_k the principal G-bundle over S^4 corresponding to $k \in \mathbb{Z}$. Let $\mathcal{C}_k(G)$ be the based moduli space of connections on P_k. Then we have a natural homotopy equivalence

$$\mathcal{C}_k(G) \simeq \Omega_k^3 G$$

where $\Omega_k^3 G$ stands for the path component of $\Omega^3 G$ corresponding to $k \in \mathbb{Z} = \pi_3(G)$. We will identify $\mathcal{C}_k(G)$ with $\Omega_k^3 G$ by this homotopy equivalence. Let $\mathcal{M}_k(G)$ be the based moduli space of instantons on P_k. Then we have a map

$$\theta_k : \mathcal{M}_k(G) \to \Omega_0^3 G$$

defined by the composite of the inclusion $\mathcal{M}_k(G) \to \Omega_k^3(G) \simeq \mathcal{C}_k(G)$ and the homotopy equivalence $\Omega_k^3 G \simeq \Omega_0 G$, the shift by $-k \in \mathbb{Z} = \pi_3(G)$.

The topology of the map θ_k was first studied by Atiyah and Jones [3], and, later, it was proved by Boyer, Hurtubise, Mann and Milgram [8], Kirwan [14] and Tian [18] that the map θ_k is a homotopy equivalence in a range, which is known as the Atiyah–Jones theorem. As a consequence of this result, Tian [18] showed that the colimit of the map θ_k yields real and symplectic Bott periodicity. However, an explicit geometric description of the homotopy equivalence is not known while Bott periodicity was given by a map explicitly defined by the commutator maps of the classical groups [6]. In [9], it is shown that the map θ_k has some relation with the commutator map of G when $k = 1$. Recall that Bott [5] also used

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the commutator maps to study the topology of loop spaces of Lie groups. Exploiting the above result of [9] in connection with the classical result of Bott [5], Kamiyama [12] studied a triple loop space analogue of generating varieties of Bott [5].

We will give a mild generalization of the above result of [9] for arbitrary k. Using this, we prove triple loop space version of the generating variety argument [5] in a sense somewhat different from [12], and also prove Bott periodicity. This yields a new proof of real and symplectic Bott periodicity. We will give applications of this result to the homotopy types of $\mathcal{M}_k(G)$.

2. Subgroups of classical groups isomorphic with SU(2)

Let G be a compact, connected, simple Lie group with a fixed isomorphism $\pi_3(G) \cong \mathbb{Z}$. Note that G acts on $\mathcal{M}_k(G)$ via the action of the basepoint free gauge group of P_k on $\mathcal{M}_k(G)$. As is shown in [9], there is an orbit of this action for $k = 1$ such that the restriction of $\theta_1 : \mathcal{M}_1(G) \to \Omega_0^3 G$ is presented by the commutator map of G. By putting additional assumption, we can prove this for arbitrary k by essentially the same way in [9] as follows.

Lemma 1. *Suppose that there exists a subgroup H of G isomorphic to* $\mathrm{SU}(2) \approx S^3$ *such that the inclusion $\iota : H \hookrightarrow G$ represents $k \in \mathbb{Z} = \pi_3(G)$. Then there exists $\omega \in \mathcal{M}_k(G)$ satisfying*:

(1) *The orbit space $G \cdot \omega$ is homeomorphic with $G/C(H)$, where $C(H)$ stands for the centralizer of H.*
(2) *Let Γ denote the composite*:

$$G/C(H) \approx G \cdot \omega \hookrightarrow \mathcal{M}_k(G) \xrightarrow{\theta_k} \Omega_0^3 G.$$

Then we have

$$\Gamma\big(gC(H)\big) \simeq g\iota(h)g^{-1}\iota(h)^{-1}$$

for $g \in G$, $h \in H$.

Proof. Let α be an asymptotically flat connection on P_k. We regard S^4 as $\mathbb{R}^4 \cup \{\infty\}$. Recall from [3] that the homotopy equivalence $\mathcal{C}_k(G) \xrightarrow{\simeq} \Omega_0^3 G$ takes $\alpha \in \mathcal{M}_k(G)$ into its 'pure gauge' $\hat{\alpha} : S^3 \to G$ at $\infty \in S^4$ normalized as $\hat{\alpha}(*) = e$, where $*$ and e are the basepoint of S^3 and unity of G, respectively. (See [3].) The action of the basepoint free gauge group of P_k is locally the conjugation by G. Then the map θ_k is G-equivariant under the action of G on $\Omega_0^3 G$ given by $g \cdot \lambda(x) = g\lambda(x)g^{-1}$ for $g \in G$, $\lambda \in \Omega_0^3 G$, $x \in S^3$.

Let P be a principal SU(2)-bundle over S^4 represented by $1 \in \mathbb{Z} \cong \pi_3(\mathrm{SU}(2))$. In [2], an asymptotically flat instanton ϖ whose pure gauge represents $1 \in \mathbb{Z} \cong \pi_3(\mathrm{SU}(3))$. Then the proof is completed by putting ω to be the push forward of ϖ by the inclusion $\iota : H \cong \mathrm{SU}(2) \to G$. □

The original form of Bott periodicity [6] is given by such a map Γ in Lemma 1 where $\mathrm{SU}(2) \approx S^3$ is replaced with $\mathrm{U}(1) \approx S^1$. On the other hand, there is known a deep relation between $\mathcal{M}_k(G)$ and Bott periodicity as in [14,17,18]. Then we expect the map Γ in Lemma 1 may yield real and symplectic Bott periodicity which has period 4. Also we expect $G/C(H)$ and Γ in Lemma 1 may yield a 3-fold loop analogue of a generating variety for a loop space of a Lie group, which is already studied by Kamiyama [12] in a slightly different sense, that is, algebras over the Kudo–Araki operations. Then we introduce a family of subgroups of the classical groups which are isomorphic with SU(2) by which we can prove the above argument.

Hereafter, we put $(\mathbf{G}, \mathbf{H}, d) = (\mathrm{Sp}, \mathrm{O}, 1), (\mathrm{SU}, \mathrm{U}, 2), (\mathrm{SO}, \mathrm{Sp}, 4)$. We will define a family of subgroups $S_{k,l}(\mathbf{G})$ of $\mathbf{G}(dk + l)$ indexed by positive integers k and non-negative integers l. Since the Lie group $\mathbf{G}(dk + l)$ must be simple, we will assume $dk + l > 4$ when $\mathbf{G} = \mathrm{SO}$.

Let $\mathbf{c} : \mathrm{O}(n) \to \mathrm{U}(n)$, $\mathbf{q} : \mathrm{U}(n) \to \mathrm{Sp}(n)$, $\mathbf{c}' : \mathrm{Sp}(n) \to \mathrm{SU}(2n)$, and $\mathbf{r} : \mathrm{U}(n) \to \mathrm{O}(2n)$ be the canonical inclusions. In order to make things clear, we write the maps $\mathbf{c}'$ and $\mathbf{r}$ explicitly as follows. Let $\mathrm{M}_n(\mathbb{K})$ be the set of all square matrices of order n over a field $\mathbb{K}$. For $A = (a_{ij})$, $B = (b_{ij}) \in \mathrm{M}_n(\mathbb{C})$ such that $A + B\mathbf{j} \in \mathrm{Sp}(n)$, we put

$$\mathbf{c}'(A + B\mathbf{j}) = \big(\mathbf{c}'(a_{ij} + b_{ij}\mathbf{j})\big)$$

where $\mathbf{c}'(a + \mathbf{j}b) = \left(\begin{smallmatrix} a & -\bar{b} \\ b & \bar{a} \end{smallmatrix}\right)$ for $a, b \in \mathbb{C}$. We also put, for $C = (c_{ij})$, $D = (d_{ij}) \in \mathrm{M}_n(\mathbb{R})$ such that $C + D\sqrt{-1} \in \mathrm{U}(n)$,

$$\mathbf{r}(C + D\sqrt{-1}) = \big(\mathbf{r}(c_{ij} + d_{ij}\sqrt{-1})\big)$$

where $\mathbf{r}(c + d\sqrt{-1}) = \left(\begin{smallmatrix} c & -d \\ d & c \end{smallmatrix}\right)$ for $c, d \in \mathbb{R}$. We denote the matrix $\left(\begin{smallmatrix} A & O \\ O & B \end{smallmatrix}\right)$ by $A \oplus B$. We consider the following family of subgroups of the classical groups isomorphic with $\mathrm{SU}(2) \approx S^3$: