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The homogeneous space G/H as an equivariant fibrant space

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ABSTRACT

In this paper we discuss some properties of equivariant fibrant spaces. It is shown that for every compact metrizable group G and its closed subgroup H, the quotient space G/H is a fibrant G-space.

0. Introduction

The general approach to the concept of *a fibrant object* is the following (cf. [5]): for a given class Σ of morphisms of a category $\mathcal{C}$, an object Y of $\mathcal{C}$ is called Σ-fibrant if for every morphism $s \in \Sigma$, $s : A \to X$, and every morphism $f : A \to Y$ there exists a morphism $F : X \to Y$ such that $F \circ s = f$. The classical fibrant objects appear in [11] for the closed model categories, where Σ is the class of trivial cofibrations.

A fibrant space in the sense of F. Cathey [8] is a Σ-fibrant object, where Σ is the class of *SSDR*-maps in the category of metrizable spaces. For example, *ANR*-spaces as well as inverse limits of *ANR*-spaces bonded by fibrations are fibrant spaces. In the present paper we consider the equivariant version of a fibrant space introduced in [6].

One of the reasons to introduce the notion of fibrant G-spaces is its role in the construction of the equivariant strong shape category in [7]. Another motive is represented by the main result of this paper (Theorem 5.1): if H is a closed subgroup of a compact metrizable group G, then the homogeneous space G/H is a fibrant G-space. In particular, all the orbits of any G-space are equivariant fibrant spaces.

It is known that every compact metrizable group G can be regarded as the inverse limit of a sequence of Lie groups bonded by fibrations, and therefore it is a fibrant space in the sense of F. Cathey. In [6] it was shown that G is also a fibrant space in the equivariant sense. In the present paper we generalize this fact.

Naturally, in order to prove our main theorem, we represent the quotient space G/H as the inverse limit of a sequence of G-*ANR*-spaces bonded by maps, which are equivariant fibrations or G-fibrations. For this, first of all, we utilize the well-known result of R.S. Palais [9, Corollary 1.6.7]: if H is a closed subgroup of a compact Lie group G, then G/H is a G-*ANR*. On the other hand, we have to prove some statements concerning special cases of G-fibrations. Proposition 2.2 and Proposition 5.5 are in fact key points in the proof of the main result.

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1. Preliminaries

The basic definitions and facts of the theory of G-spaces can be found in [4]. In this section we shall recall some of them.

The letters G and e will denote a compact Hausdorff group and its unit element respectively.

A *G-space* is a topological space X together with a fixed continuous left action $G \times X \to X$, $(g, x) \mapsto g \cdot x$, of G on X; $g \cdot x$ also will be denoted simply by gx. A subset $A \subset X$ is *invariant* or G-invariant if $ga \in A$ for all $a \in A$ and $g \in G$. A *G-map* or an *equivariant* map $f : X \to Y$ between G-spaces is a continuous map satisfying $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and $x \in X$.

Given a closed subgroup H of G, we can consider the *homogeneous space* G/H which is the quotient space of G consisting of left cosets xH for $x \in G$. Clearly G/H is a G-space with the action $g \cdot (xH) = (gx)H$ and the natural projection $\pi : G \to G/H$, $x \mapsto xH$ is a G-map.

Let N be a closed normal subgroup of G and let X be a G-space. For every point $x \in X$, the set $N(x) = \{nx \mid n \in N\}$ is called the *N-orbit* of x. The set $X/N = \{N(x) \mid x \in X\}$, which is a quotient space of X, is called the *N-orbit space* of X. The space X/N is a G/N-space with the action $gN \cdot N(x) = N(gx)$. Every G-map $f : X \to Y$ induces a G/N-map $f/N : X/N \to Y/N$ given by $(f/N)(N(x)) = N(f(x))$.

For any H-space X, where H is a closed subgroup of G, the *twisted product* $G \times_H X$ is defined as the orbit space of the H-space $G \times X$ with respect to the action $h \cdot (g, x) = (gh^{-1}, hx)$. The twisted product $G \times_H X$ is a G-space with the action given by $g' \cdot [g, x] = [g'g, x]$, where $[g, x]$ denotes the H-orbit of $(g, x) \in G \times X$. Moreover, every H-map $f : X \to Y$ induces a G-map $G \times_H f : G \times_H X \to G \times_H Y$ defined by $(G \times_H f)[g, x] = [g, f(x)]$.

Let X and Y be G-spaces. A homotopy $F : X \times I \to Y$, where $I = [0, 1]$, is called a *G-homotopy*, if $F(gx, t) = gF(x, t)$ for all $g \in G$, $x \in X$ and $t \in I$. Thus F is a G-map, considering $X \times I$ as a G-space with the action $g \cdot (x, t) = (gx, t)$. Note also that for every $t \in I$ the map $F_t : X \to Y$, $x \mapsto F(x, t)$ is a G-map.

Let A be an invariant subset of X. A G-homotopy $F : X \times I \to Y$ is *relative to* A when $F(a, t) = F(a, 0)$ for all $a \in A$ and $t \in I$. Two G-maps $f_0, f_1 : X \to Y$ such that $f_0|_A = f_1|_A$ are called *G-homotopic relative to* A, written $f_0 \simeq_G f_1$ (*rel* A), if $f_0 = F_0$ and $f_1 = F_1$ for some G-homotopy $F : X \times I \to Y$ relative to A.

By *G-ANR* it is denoted a *G-equivariant absolute neighborhood retract* for all G-metrizable spaces (see, for instance, [1–3] for the equivariant theory of retracts). It is known [2] that a metrizable G-space Y is a *G-ANR* if and only if it is *G-ANE*, in other words, it has the following extension property: for any G-map $f : A \to Y$, where A is a closed invariant subset of a metrizable G-space X, there exists a G-map $\overline{f} : U \to Y$ such that $\overline{f}|_A = f$, where U is some invariant neighborhood of A in X.

2. Regular *G*-fibrations

By G-fibration we shall mean an equivariant version of Hurewicz fibration: a G-map $p : E \to B$ is a *G-fibration* if it has the right lifting property with respect to the G-embedding $X \hookrightarrow X \times I$, $x \mapsto (x, 0)$, where X is arbitrary G-space.

We say that a G-map $p : E \to B$ is a *regular G-fibration*, if for every closed invariant subset A of a G-space X and every commutative diagram of G-maps

$$\begin{array}{ccc} X & \xrightarrow{f} & E \\ {\scriptstyle \delta_0}\downarrow & & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[F]{} & B \end{array}$$

where $\delta_0(x) = (x, 0)$ and F is a G-homotopy relative to A, there exists a G-homotopy $\widetilde{F} : X \times I \to E$ also relative to A such that $\widetilde{F} \circ \delta_0 = f$ and $p \circ \widetilde{F} = F$.

Proposition 2.1. *Let $p : E \to B$ be a G-fibration. If B is a metrizable G-space, then p is a regular G-fibration.*

The proof of this statement is analogous to the well-known proof for the non-equivariant case and involves the following fact [2, Proposition 5]: every metrizable G-space B admits a compatible *invariant* metric, that is a metric d for which $d(gx, gy) = d(x, y)$ for all $g \in G$ and $x, y \in B$.

The following statement will be essentially used in the proof of the main result of the paper.

Proposition 2.2. *Let H be a closed subgroup of a metrizable group G. If H is a compact Lie group, then the natural projection $\pi : G \to G/H$ is a regular G-fibration.*