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C-spaces and matrix of infinite-dimensionality

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ABSTRACT

We investigate classes (m,n)-C which are intermediate between the class S-wid of weakly infinite-dimensional spaces in the sense of Smirnov and the class S-∞-C of finite C-spaces in the sense of Borst. We find relationships between classes (m_1,n_1)-C and (m_2,n_2)-C. It allows us to construct a matrix of infinite-dimensionality.

0. Introduction

Property C for metric spaces was introduced by Haver in 1973 [12], who proved that every locally contractible metric space which can be represented as a union of countably many compacta with property C is an ANR-space. A topological definition of C-spaces was proposed by Addis and Gresham in 1978 [1]. Recall their definition.

1. Definition. A topological space X is said to be a *C-space* (notation: $X \in C$) if for every sequence (u_i), $i \in \mathbb{N}$, of open covers of X there exists a sequence (v_i) of disjoint families of open subsets of X such that each family v_i is a refinement of u_i, and the collection of families v_i is a cover of X.

It has become clear that C-spaces play an important role in the theory of cell-like maps [2], in the study of selections of multivalued maps [11,14,15], and in other areas of topology.

The nerves $N(v_i)$ of disjoint families v_i from Definition 1 are zero-dimensional simplicial complexes. One can consider an arbitrary class $\mathcal{G}$ of simplicial complexes instead of the class of zero-dimensional complexes. In such a manner in [9] there were introduced classes of $\mathcal{G}$-C-spaces, S-$\mathcal{G}$-C-spaces (Definition 2.8), and classes m-$\mathcal{G}$-C-spaces, and S-m-$\mathcal{G}$-C-spaces (Definition 2.11). In Section 2 we recall some properties of these classes of spaces from [9].

In Section 3 we study the classes (m,n)-$C \equiv S$-m-$\mathcal{G}(n-1)$-C, where $\mathcal{G}(k)$ is the class of all finite simplicial complexes G with $\dim G \leqslant k$. Theorem 3.1 states that (m,n)-$C \subset (km,kn)$-C for positive integers k,m,n with $n \leqslant m$. Theorem 3.2 contains an "opposite" inclusion: (m^k,kn)-$C \subset (m,n)$-C. Main result of Section 3 is Theorem 3.3 which asserts that the classes $(n+1,n)$-C coincide with the class S-wid of all weakly infinite-dimensional spaces in the sense of Smirnov. As a corollary, we get

$$(m,n)\text{-}C = S\text{-wid}$$

for every m with $n+1 \leqslant m \leqslant 2n$. It allows us to construct a matrix of infinite-dimensionality in Section 4.

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Section 1 contains an auxiliary information. An additional information one can find in [5]. All spaces are assumed to be normal T_1. The symbol $|A|$ stands for the cardinality of a set A. If A is a subset of a space X, then $[A] \equiv [A]_X$ denotes the closure of A in X. By $\bigsqcup$ we denote a union of disjoint sets.

1. Preliminaries

By $\operatorname{cov}(X)$ we denote the set of all open covers of a space X. The set of all finite open covers of X is denoted by $\operatorname{cov}_\infty(X)$ and $\operatorname{cov}_m(X)$ stands for the set of all open covers of X consisting of $\leqslant m$ members.

Let u and v be families of subsets of a set X. They say that v *refines* u (v is a *refinement* of u) if each $V \in v$ is contained in some $U \in u$. A family v *combinatorially refines* u (v is a *combinatorial refinement* of u) if there exists an injection $i : v \to u$ such that $V \subset i(V)$ for each $V \in v$.

For a simplicial complex G by $v(G)$ we denote the set of all its vertices. By $\operatorname{Fin} S$ we denote the set of all non-empty finite subsets of a set S. Let u be a family of arbitrary sets and let $u_0 = \{U \in u\colon\ U \neq \emptyset\}$. The *nerve* $N(u)$ of the family u is a simplicial complex such that $v(N(u)) = \{a_U\colon\ U \in u_0\}$ and a set $\Delta \in \operatorname{Fin} v(N(u))$ is a simplex of $N(u)$ if and only if $\bigcap\{U\colon\ a_U \in \Delta\} \neq \emptyset$.

By the *order* of a family u of sets we mean the largest n such that u contains n sets with a non-empty intersection. If no such integer exists, we say that u has order ∞. The order of u is denoted by $\operatorname{ord} u$. Clearly,

$$\operatorname{ord} u \leqslant n \quad \Leftrightarrow \quad \dim N(u) \leqslant n-1; \tag{1.1}$$

$$\operatorname{ord} u \leqslant 1 \quad \Leftrightarrow \quad u \text{ is a disjoint family.} \tag{1.2}$$

1.1. Proposition. *Let $u \in \operatorname{cov}_\infty(X)$ and let $\operatorname{ord} u = n$. Then there exist families v_i, $i = 1, \dots, n$, consisting of open subsets of X such that v_i combinatorially refine u, $\bigcup\{v_i\colon\ 1, \dots, n\} \in \operatorname{cov}(X)$, and $\operatorname{ord} v_i \leqslant 1$.*

1.2. Definition. Let $u = \{U_1, \dots, U_n\} \in \operatorname{cov}(X)$ and let a family v refine u. Define a new family v_1 in the following way:

$$V_i^1 = \bigcup\{V \in v\colon\ V \subset U_i,\ j < i \Rightarrow V \not\subset U_j\}; \quad v_1 = \{V_1^1, \dots, V_n^1\}.$$

The family v_1 is called an *integration* of v with respect to u.

1.3. Proposition. *If v_1 is an integration of v with respect to u, then*

v_1 combinatorially refines u;

$\operatorname{ord} v_1 \leqslant \operatorname{ord} v$;

$\bigcup v_1 = \bigcup v$, *in particular,* $v \in \operatorname{cov}(X) \Rightarrow v_1 \in \operatorname{cov}(X)$.

Recall that a space X is said to be *weakly infinite-dimensional in the sense of Smirnov* (notation: $X \in S$-wid) if for every sequence $\Phi_i = \{F_1^i, F_2^i\}$, $i \in \mathbb{N}$, of pairs of disjoint closed subsets of X there exist partitions P_i between F_1^i and F_2^i and $n \in \mathbb{N}$ such that $\bigcap_{i=1}^n P_i = \emptyset$.

By wid we denote the class of all *weakly infinite-dimensional spaces in the sense of Alexandroff* (notation: $X \in$ wid). One can get its definition if the requirement $\bigcap_{i=1}^n P_i = \emptyset$ from the definition of S-wid-spaces replace by the requirement $\bigcap_{i=1}^\infty P_i = \emptyset$. For compact (and even countably compact) spaces the classes S-wid and wid coincide.

1.4. Theorem. *([7,13]) For an arbitrary space X we have*

$$X \in S\text{-wid} \quad \Leftrightarrow \quad \beta X \in \text{wid}.$$

For topological spaces $X_1, \dots, X_n$, let $B(X_1, \dots, X_n)$ be a subset of $\prod_{i=1}^n \operatorname{cone} X_i$ which is defined as follows

$$B(X_1, \dots, X_n) \equiv B = B_1 \cup \dots \cup B_n,$$

where

$$B_i = X_i \times \prod_{j \neq i} \operatorname{cone} X_j.$$

1.5. Definition. ([8]) Let $\mathcal{R}$ be a class of ANR-compacta and let

$$f_i : X \to \operatorname{cone} R_i, \quad R_i \in \mathcal{R},\ i = 1, \dots, n,$$

be maps. The family $\{f_1, \dots, f_n\}$ is said to be $\mathcal{R}$-*inessential* if the map

$$f = f_1 \Delta \cdots \Delta f_n|_{f^{-1}B(R_1, \dots, R_n)} : f^{-1}B(R_1, \dots, R_n) \to B(R_1, \dots, R_n)$$

extends over X.