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Finite unions of weak $\bar{\theta}$-refinable spaces and products of ordinals ☆

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ABSTRACT

In this note, we show that if X is the union of a finite collection $\{X_i: i = 1, \dots, k\}$ of weak $\bar{\theta}$-refinable subspaces and $e(X) = \omega$, then X is a Lindelöf space. We also show that the product of two ordinals is dually discrete. The last conclusion gives a positive answer to a question of Alas, Junqueira and Wilson.

0. Introduction

The notion of D-space was introduced by van Douwen (cf. [9]). A *neighborhood assignment* for a space X is a function ϕ from X to the topology of the space X, such that $x \in \phi(x)$ for any $x \in X$. A space X is called a *D-space*, if for any neighborhood assignment ϕ for X there exists a closed discrete subspace D of X such that $X = \bigcup\{\phi(d): d \in D\}$ (cf. [10]). In [10], it was proved that a finite product of the Sorgenfrey line is a D-space. By results of [6], we know that all semi-stratifiable spaces are D-spaces. So we know that all metrizable spaces and all Moore spaces are D-spaces. In [7], Buzyakova proved that every strong Σ-space is a D-space. In 2004 (cf. [3]), Arhangel'skii proved that if X is the union of a finite collection of spaces with a point-countable base, then X is a D-space. In 2008, Peng proved that if X is the union of a finite collection of Moore spaces, then X is a D-space (cf. [16]). The idea of D-spaces was then further developed in recent years, the properties of αD-spaces and dually discrete spaces were discussed.

The concept of an αD-space was introduced in [5]. A space X is an *αD-space* if for each closed subset F of X and each open covering $\mathcal{U}$ of X there exists a locally finite in F subset A of F and a mapping ϕ of A into $\mathcal{U}$ such that $a \in \phi(a)$ for each $a \in A$, and the family $\phi(A) = \{\phi(a): a \in A\}$ covers F. By results of [6], we know that every subparacompact space is an αD-space. In [5], it was proved that if a regular space X is the union of a finite collection of paracompact subspaces then X is an αD-space. In [3], it was proved that if X is the union of a finite collection of subparacompact subspaces then X is an αD-space. Peng also proved that X is an αD-space if X is the union of finite collection of θ-refinable subspaces (cf. [17, Theorem 10]). We also know that very $\delta\theta$-refinable space is an αD-space (cf. [4, Theorem 1.15]).

The concept of weak $\bar{\theta}$-refinable spaces was introduced in [19]. It was proved that every θ-refinable space is a weak $\bar{\theta}$-refinable space and a weak $\bar{\theta}$-refinable space X with countable extent (every countable closed discrete subspace of X is

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countable, usually be denoted by $e(X) = \omega$) is Lindelöf (cf. [19, Theorem 3.3]). By results of [14], we know that every weak $\bar{\theta}$-refinable space is an αD-space. The following question is open: Is the union of a finite collection $\{X_i\colon\ i \leqslant n\}$ of weak $\bar{\theta}$-refinable subspaces an αD-space? We know that every countably compact αD-space is compact. Although the above question is not answered in this note, we show that if X is the union of a finite collection $\{X_i\colon\ i \leqslant n\}$ of weak $\bar{\theta}$-refinable subspaces and $e(X) = \omega$, then X is Lindelöf.

The concept of dually discrete spaces was introduced in [13]. A space X is called *dually discrete*, if for any neighborhood assignment ϕ for X there exists a discrete subspace D of X such that $X = \bigcup\{\phi(d)\colon\ d \in D\}$. Every D-space is dually discrete. There has been much work on dual properties (cf. [1,2,8,13]).

In [8], it was proved that every ordinal is dually discrete. In [1], it was proved that a finite product of regular cardinals is dually discrete. The following problem appeared in [1]: Is the product of two ordinals (hereditarily) dually discrete? In this note, we show that a finite product of ordinals is dually discrete.

All the spaces in this note are assumed to be T_1-spaces. The set of all natural numbers is denoted by N, ω is $N \cup \{0\}$. In notation and terminology we will follow [11].

1. On finite unions of weak $\bar{\theta}$-refinable spaces

Let X be a space and let $\mathcal{U}$ be a family of open subsets of X. We denote $ord(x,\mathcal{U}) = |\{V\colon\ x \in V \text{ and } V \in \mathcal{U}\}|$.

Lemma 1.1. *(Cf. [18, Lemma 3].) Let $\mathcal{U} = \{U_\lambda\colon\ \lambda < \eta\}$ be an open cover of X and $X_n = \{x \in X\colon\ ord(x,\mathcal{U}) \leqslant n\}$ for each $n \in N$. Then X_n is a closed subset of X and $\mathcal{F}_n = \{E(\lambda_1,\dots,\lambda_n)\colon\ \lambda_1 < \cdots < \lambda_n < \eta\}$ is a discrete cover of the subspace $X_n \setminus X_{n-1}$ for each $n \geqslant 1$, where $E(\lambda_1,\dots,\lambda_n) = \bigcap\{U_{\lambda_i} \cap (X_n \setminus X_{n-1})\colon\ i \leqslant n\}$ and $X_0 = \emptyset$.*

Lemma 1.2. *(Cf. [16, Lemma 1].) Suppose $X = \bigcup\{X_i\colon\ 1 \leqslant i \leqslant n\}$ and $\mathcal{F}$ is a locally finite family of subsets of X_i for some $i \leqslant n$ and $A = \{x\colon\ x \in X$ and $\mathcal{F}$ is not locally finite at $x\}$. Then A is a closed subset of X and $A \subset X \setminus X_i$.*

Lemma 1.3. *Let $X = X_1 \cup X_2$ and $e(X) = \omega$ and let $Y \subset X_1$ be a closed subspace of X_1. Let $\mathcal{U}$ be an open cover of X and let $\mathcal{V} = \bigcup\{\mathcal{V}_n\colon\ n \in N\}$ be an open refinement of $\mathcal{U}$, such that $Y \subset \bigcup \mathcal{V}_n$ for each $n \in N$ and for each $y \in Y$ there exists some $n_y \in N$ such that $1 \leqslant ord(x, \mathcal{V}_{n_y}) < \omega$. If each closed subspace F of X which is contained in $X_2 \setminus X_1$ is Lindelöf, then there is a countable subfamily $\mathcal{U}_Y \subset \mathcal{U}$ such that $Y \subset \bigcup \mathcal{U}_Y$.*

Proof. For each $m \in N$ and $n \in N$, let $F_{mn} = \{x\colon\ x \in Y$ and $ord(x, \mathcal{V}_m) \leqslant n\}$. By Lemma 1.1, we know that F_{mn} is a closed subspace of Y and $F_{m(n+1)} \setminus F_{mn} = \bigcup \mathcal{F}_{m(n+1)}$, where $\mathcal{F}_{m(n+1)}$ is a closed and open (in $F_{m(n+1)} \setminus F_{mn}$) cover of $F_{m(n+1)} \setminus F_{mn}$ and for each $F \in \mathcal{F}_{m(n+1)}$ there is some $V_F \in \mathcal{V}_m$ such that $F \subset V_F$.

For each $m \in N$, the set $\mathcal{F}_{m1}$ is a closed discrete family of subsets of Y and Y is a closed in X_1, so $\mathcal{F}_{m1}$ is a discrete family in X_1. Let $A_{m1} = \{x\colon\ x \in X$ and $\mathcal{F}_{m1}$ is not locally finite at $x\}$. Then A_{m1} is a closed subset of X and $A_{m1} \subset X_2 \setminus X_1$ by Lemma 1.2. Thus A_{m1} is a closed Lindelöf subspace of X. Thus there exists a countable family $\mathcal{U}_1^* \subset \mathcal{U}$ such that $A_{m1} \subset \bigcup \mathcal{U}_1^*$. Thus $F_{m1} \setminus \bigcup \mathcal{U}_1^* = \bigcup\{F \setminus \bigcup \mathcal{U}_1^*\colon\ F \in \mathcal{F}_{m1}\}$. The family $\{F \setminus \bigcup \mathcal{U}_1^*\colon\ F \in \mathcal{F}_{m1}\}$ is locally finite in X, so $\{F \setminus \bigcup \mathcal{U}_1^*\colon\ F \in \mathcal{F}_{m1}\}$ is countable since $e(X) = \omega$.

For each $F \in \mathcal{F}_{m1}$, there exists some $U_F \in \mathcal{U}$ such that $F \setminus \bigcup \mathcal{U}_1^* \subset U_F$ if $F \setminus \bigcup \mathcal{U}_1^* \neq \emptyset$. Thus there exists a countable subfamily $\mathcal{U}_1^{**}$ of $\mathcal{U}$ such that $\bigcup\{F \setminus \bigcup \mathcal{U}_1^*\colon\ F \in \mathcal{F}_{m1}\} \subset \bigcup \mathcal{U}_1^{**}$. If $\mathcal{U}_1 = \mathcal{U}_1^* \cup \mathcal{U}_1^{**}$ then $\mathcal{U}_1 \subset \mathcal{U}$ and $|\mathcal{U}_1| \leqslant \omega$ such that $F_{m1} \subset \bigcup \mathcal{U}_1$.

For each $n \in N$, we assume that there is a countable subfamily $\mathcal{U}_k$ of $\mathcal{U}$ such that $F_{mk} \subset \bigcup\{\bigcup \mathcal{U}_j\colon\ j \leqslant k\}$ for each $k \leqslant n$.

For $n+1 \in N$, we have $F_{m(n+1)} \setminus F_{mn} = \bigcup \mathcal{F}_{m(n+1)}$ and $F_{mn} \subset \bigcup\{\bigcup \mathcal{U}_j\colon\ j \leqslant n\}$. If $\mathcal{F}^*_{m(n+1)} = \{F \setminus (\bigcup\{\bigcup \mathcal{U}_j\colon\ j \leqslant n\})\colon F \in \mathcal{F}_{m(n+1)}\}$ then $\mathcal{F}^*_{m(n+1)}$ is a discrete family of subsets of Y. So $\mathcal{F}^*_{m(n+1)}$ is also a discrete family in X_1. Let $A_{m(n+1)} = \{x\colon\ x \in X$ and $\mathcal{F}^*_{m(n+1)}$ is not locally finite at $x\}$. Thus $A_{m(n+1)}$ is a closed subset of X and $A_{m(n+1)} \subset X_2 \setminus X_1$ by Lemma 1.2. So $A_{m(n+1)}$ is a closed Lindelöf subspace of X. Thus $A_{m(n+1)}$ can be covered by a countable subfamily $\mathcal{U}^*_{n+1}$ of $\mathcal{U}$. So $\mathcal{F}^{**}_{m(n+1)} = \{B \setminus \bigcup \mathcal{U}^*_{n+1}\colon\ B \in \mathcal{F}^*_{m(n+1)}\}$ is a locally finite family in X. Since $e(X) = \omega$, we have $|\mathcal{F}^{**}_{m(n+1)}| \leqslant \omega$. For each $C \in \mathcal{F}^{**}_{m(n+1)}$ there is some $U_C \in \mathcal{U}$ such that $C \subset U_C$. So there is a countable subfamily $\mathcal{U}^{**}_{n+1}$ of $\mathcal{U}$ such that $\bigcup \mathcal{F}^{**}_{m(n+1)} \subset \bigcup \mathcal{U}^{**}_{n+1}$. If $\mathcal{U}_{n+1} = \mathcal{U}^*_{n+1} \cup \mathcal{U}^{**}_{n+1}$ then $\mathcal{U}_{n+1} \subset \mathcal{U}$ and $|\mathcal{U}_{n+1}| \leqslant \omega$ such that $F_{m(n+1)} \subset \bigcup\{\bigcup \mathcal{U}_j\colon\ j \leqslant n+1\}$. Thus $\bigcup\{F_{mn}\colon\ n \in N\}$ can be covered by a countable subfamily of $\mathcal{U}$ by induction.

The set $Y = \bigcup\{F_{mn}\colon\ m \in N$ and $n \in N\}$, thus there exists some countable subfamily $\mathcal{U}_Y$ of $\mathcal{U}$ such that $Y \subset \bigcup \mathcal{U}_Y$. □

Definition 1.4. (Cf. [19].) A space X is called *weak $\bar{\theta}$-refinable* if for any open cover $\mathcal{U}$ of X there is an open refinement $\mathcal{V} = \bigcup\{\mathcal{V}_i\colon\ i \in N\}$ such that:

(1) $\{\bigcup \mathcal{V}_i\colon\ i \in N\}$ is point-finite.
(2) For any $x \in X$ there is some $i \in N$ such that $1 \leqslant ord(x, \mathcal{V}_i) < \infty$.

The refinement $\mathcal{V} = \bigcup\{\mathcal{V}_i\colon\ i \in N\}$ of $\mathcal{U}$ is said to be a *weak $\bar{\theta}$-refinement* of $\mathcal{U}$.