

On the categorical meaning of Hausdorff and Gromov distances, I

Andrei Akhmediani^{a,1}, Maria Manuel Clementino^{b,2}, Walter Tholen^{c,*,1}

^a Oxford University Computing Laboratory, Oxford OX1 3QD, United Kingdom

^b Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal

^c Department of Mathematics and Statistics, York University, Toronto, ON M3J 1P3, Canada

ARTICLE INFO

Article history:

Received 5 January 2009

Received in revised form 16 June 2009

Accepted 18 June 2009

MSC:

primary 18E40

secondary 18A99

Keywords:

Lawvere metric space

$\mathcal{V}$ -category

$\mathcal{V}$ -module

Hausdorff metric

Gromov metric

ABSTRACT

Hausdorff and Gromov distances are introduced and treated in the context of categories enriched over a commutative unital quantale $\mathcal{V}$. The Hausdorff functor which, for every $\mathcal{V}$ -category X , provides the powerset of X with a suitable $\mathcal{V}$ -category structure, is part of a monad on $\mathcal{V}\text{-Cat}$ whose Eilenberg–Moore algebras are order-complete. The Gromov construction may be pursued for any endofunctor K of $\mathcal{V}\text{-Cat}$. In order to define the Gromov “distance” between $\mathcal{V}$ -categories X and Y we use $\mathcal{V}$ -modules between X and Y , rather than $\mathcal{V}$ -category structures on the disjoint union of X and Y . Hence, we first provide a general extension theorem which, for any K , yields a lax extension $\bar{K}$ to the category $\mathcal{V}\text{-Mod}$ of $\mathcal{V}$ -categories, with $\mathcal{V}$ -modules as morphisms.

© 2009 Elsevier B.V. All rights reserved.

1. Introduction

The Hausdorff metric for (closed) subsets of a (compact) metric space has been recognized for a long time as an important concept in many branches of mathematics, and its origins reach back even beyond Hausdorff [10], to Pompeiu [15]; for a modern account, see [2]. It has gained renewed interest through Gromov’s work [9]. The Gromov–Hausdorff distance of two (compact) metric spaces is the infimum of their Hausdorff distances after having been isometrically embedded into any common larger space. There is therefore a notion of convergence for (isometry classes of compact) metric spaces which has not only become an important tool in analysis and geometry, but which has also provided the key instrument for the proof of Gromov’s existence theorem for a nilpotent subgroup of finite index in every finitely-generated group of polynomial growth [8].

By interpreting the (non-negative) distances $d(x, y)$ as $\text{hom}(x, y)$ and, hence, by rewriting the conditions

$$0 \geq d(x, x), \quad d(x, y) + d(y, z) \geq d(x, z) \quad (*)$$

as

$$k \rightarrow \text{hom}(x, x), \quad \text{hom}(x, y) \otimes \text{hom}(y, z) \rightarrow \text{hom}(x, z),$$

Lawvere [13] described metric spaces as categories enriched over the (small and “thin”) symmetric monoidal-closed category $\mathbb{P}_+ = ([0, \infty], \geq, +, 0)$, and in his paper [14] as well as in the Foreword of the electronic “reprint” of [13] he suggested

* Corresponding author.

E-mail addresses: andrei.akhmediani@comlab.ox.ac.uk (A. Akhmediani), mmc@mat.uc.pt (M.M. Clementino), tholen@mathstat.yorku.ca (W. Tholen).

¹ The authors acknowledge partial financial assistance from NSERC.

² The author acknowledges financial support from the Center of Mathematics of the University of Coimbra/FCI.

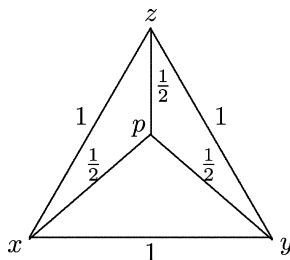


Fig. 1.

that the Hausdorff and Gromov metrics should be developed for an arbitrary symmetric monoidal-closed category $(\mathcal{V}, \otimes, k)$. In this paper we present notions of Hausdorff and Gromov distance for the case that $\mathcal{V}$ is “thin”. Hence, we replace $\mathbb{P}_+$ by a commutative and unital *quantale* $\mathcal{V}$, that is: by a complete lattice which is also a commutative monoid $(\mathcal{V}, \otimes, k)$ such that the binary operation $\otimes$ preserves suprema in each variable. Put differently, we try to give answers to questions of the type: which structure and properties of the (extended) non-negative real half-line allow for a meaningful treatment of Hausdorff and Gromov distances, and which are their appropriate carrier sets? We find that the guidance provided by enriched category theory [12] is almost indispensable for finding satisfactory answers, and that so-called *(bi-)modules* (or *distributors*) between $\mathcal{V}$ -categories provide an elegant tool for the theory which may easily be overlooked without the categorical environment. Hence, our primary motivation for this work is the desire for a better understanding of the true essentials of the classical metric theory and its applications, rather than the desire for giving merely a more general framework which, however, may prove to be useful as well.

Since $(*)$ isolates precisely those conditions of a metric which lend themselves naturally to the hom interpretation, a discussion of the others seems to be necessary at this point; these are:

- $d(x, y) = d(y, x)$ (*symmetry*),
- $x = y$ whenever $d(x, y) = 0 = d(y, x)$ (*separation*),
- $d(x, y) < \infty$ (*finiteness*).

With the distance of a point x to a subset B of the metric space $X = (X, d)$ be given by $d(x, B) = \inf_{y \in B} d(x, y)$, the *non-symmetric Hausdorff distance* from a subset A to B is defined by

$$Hd(A, B) = \sup_{x \in A} d(x, B),$$

from which one obtains the classical *Hausdorff distance*

$$H^s d(A, B) = \max\{Hd(A, B), Hd(B, A)\}$$

by *enforced* symmetrization. But not only symmetry, but also separation and finiteness get lost under the rather natural passage from d to Hd . (If one thinks of $d(x, B)$ as the travel time from x to B , then $Hd(A, B)$ may be thought of as the time needed to evacuate everyone living in the area A to the area B .) In order to save them one usually restricts the carrier set from the entire powerset PX to the closed subsets of X (which makes $H^s d$ separated), or even to the non-empty compact subsets (which guarantees also finiteness). As in [11], we call a $\mathbb{P}_+$ -category an *L-metric space*, that is a set X equipped with a function $d : X \times X \rightarrow [0, \infty]$ satisfying $(*)$; a $\mathbb{P}_+$ -functor $f : (X, d) \rightarrow (X', d')$ is a non-expansive map, e.g. a map $f : X \rightarrow X'$ satisfying $d'(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$. That the underlying-set functor makes the resulting category *Met* *topological* over *Set* (see [7]) provides further evidence that properties $(*)$ are fundamental and are better considered separately from the others, even though symmetry (as a coreflective property) would not obstruct topologicity. But inclusion of (the reflective property of) separation would, and inclusion of (the neither reflective nor coreflective property of) finiteness would make for an even poorer categorical environment.

While from the categorical perspective symmetry seems to be artificially superimposed on the Hausdorff metric, it does play a crucial role for the Gromov distance, which becomes evident already when we look at the most elementary examples. Initially nothing prevents us from considering arbitrary *L-metric spaces* X, Y and putting

$$GH(X, Y) = \inf_Z Hd_Z(X, Y),$$

where Z runs through all *L-metric spaces* Z into which both X and Y are isometrically embedded. But for $X = \{p\}$ a singleton set and $Y = \{x, y, z\}$ three equidistant points, with all distances 1, say, for every $\varepsilon > 0$ we can make $Z = X \sqcup Y$ a (proper) metric space, with $d(p, x) = d(x, p) = \varepsilon$ and all other non-zero distances 1. Then $Hd_Z(X, Y) = \varepsilon$, and $GH(X, Y) = 0$ follows. One has also $GH(Y, X) = 0$ but here one needs non-symmetric (but still separated) structures: put $d(x, p) = d(y, p) = d(z, p) = \varepsilon$, but let the reverse distances be 1. Hence, even a *posteriori* symmetrization leads to a trivial distance between non-isomorphic spaces. However, there are two ways of *a priori* symmetrization which both yield the intuitively desired result $\frac{1}{2}$ for the Gromov distance in this example (see Fig. 1): One way is by restricting the range of

Download English Version:

<https://daneshyari.com/en/article/4660587>

Download Persian Version:

<https://daneshyari.com/article/4660587>

[Daneshyari.com](https://daneshyari.com)