Topology and its Applications 157 (2010) 1296–1301

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Universal free G-spaces

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ARTICLE INFO

Article history:
Received 28 October 2008
Received in revised form 11 March 2009
Accepted 27 March 2009

MSC:
54H150
54B05
54D35

Keywords:
Universal free G-space
Join
Orbit space
Cross-section

ABSTRACT

For a compact Lie group G, we prove the existence of a universal G-space in the class of all paracompact (respectively, metrizable, and separable metrizable) free G-spaces. We show that such a universal free G-space cannot be compact.

1. Introduction

By a G-space we mean a topological space X together with a fixed continuous action $(g, x) \mapsto gx$ of a topological group G on X.

In this paper we are mostly interested in *free G-spaces*, where the acting group G is compact Lie. Recall that a G-space X is free if for every $x \in X$ the equality $gx = x$ implies $g = e$, the unity of G. A G-space U is called universal for a given class of G-spaces G–$\mathcal{P}$ if $U \in G$–$\mathcal{P}$ and U contains as a G-subspace a G-homeomorphic copy of any G-space X from the class G–$\mathcal{P}$.

In [2] it is proved that if G is a compact Lie group then there exists a locally compact, noncompact free G-space which is universal in the class of all Tychonoff free G-spaces.

On the other hand, in [1] it was established that for any integer $n \geqslant 0$, infinite cardinal number τ, and a compact Lie group G with $\dim G \leqslant n$, there exists a compact free G-space $\mathcal{F}^n_\tau$ which is universal in the class of all paracompact free G-spaces X of weight $\mathrm{w}\, X \leqslant \tau$ and dimension $\dim X \leqslant n$. As is shown in Example 3.9, this result is no longer true without the finite dimensionality restriction (we stress the compactness of the universal free G-space $\mathcal{F}^n_\tau$ in this setting).

In this connection it is natural to ask the following two questions:

Question 1. Does there exist a universal free G-space in the class of all paracompact (respectively, metrizable) free G-spaces of a given infinite weight τ?

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0166-8641/$ – see front matter
doi:10.1016/j.topol.2009.03.055

Question 2. Does there exist a universal free G-space in the class of all compact free G-spaces of a given infinite weight τ?

The purpose of this paper is to answer (in the positive) Question 1 while Question 2 still remains open.

2. Preliminaries

Throughout the paper all topological spaces and topological groups are assumed to be Tychonoff (= completely regular and Hausdorff). All equivariant or G-maps are assumed to be continuous.

The letter "G" will always denote a compact Lie group. By e we always will denote the unity of the group G.

The basic ideas and facts of the theory of G-spaces or topological transformation groups can be found in G. Bredon [3] and R. Palais [8].

For the convenience of the reader we recall, however, some more special definitions and facts below.

By an action of the group G on a space X we mean a continuous map $(g, x) \mapsto gx$ of the product $G \times X$ into X such that $(gh)x = g(hx)$ and $ex = x$ whenever $x \in X$ and $g, h \in G$. A space X together with a fixed action of the group G is called a G-space.

A continuous map $f : X \to Y$ of G-spaces is called an equivariant map or, for short, a G-map, if $f(gx) = gf(x)$ for every $x \in X$ and $g \in G$. If G acts trivially on Y then we use the term "invariant map" instead of "equivariant map". By a G-embedding we shall mean a topological embedding $X \hookrightarrow Y$ which is a G-map.

A G-space X is called free if for every $x \in X$ the equality $gx = x$ implies $g = e$.

For a subset $S \subset X$ and a subgroup $H \subset G$, $H(S)$ denotes the H-saturation of S, i.e., $H(S) = \{hs \mid h \in H,\ s \in S\}$. In particular $G(x)$ denotes the G-orbit $\{gx \in X | g \in G\}$ of x. If $H(S) = S$ then S is said to be an H-invariant set, or simply, an H-set. The set X/G of all G-orbits endowed with the quotient topology is called the G-orbit space. Often we shall use the term "invariant set" for a "G-invariant set".

In the sequel we will consider the acting group G itself as a G space endowed with the action induced by left translations.

If X and Y are G-spaces then $X \times Y$ will always be regarded as a G-space equipped by the diagonal action of G. If one of the G-spaces X and Y is free then, clearly, so is their product $X \times Y$.

Let us recall also the well-known and important definition of a local cross-section:

Definition 2.1. A subset S of a G-space X is called a local cross-section in X, if:

(1) the saturation $G(S)$ is open in X,
(2) if $g \in G \setminus \{e\}$ then $gS \cap S = \emptyset$,
(3) S is closed in $G(S)$.

The saturation $G(S)$ will be said to be a tubular set. If in addition $G(S) = X$ then we say that S is *a global* cross-section of X.

One of the basic results of the theory of topological transformation groups is the following result of A.M. Gleason [5, Theorem 3.3] about the existence of local cross-sections.

Theorem 2.2. *Let G be a compact Lie group, X a free G-space and $x \in X$ a point. Then there exists a local cross-section S in X such that $x \in S$.*

This result was further generalized to one of the most important results of the theory of transformation groups known as the Slice Theorem (see e.g., [8, Theorem 1.7.18] or [3, Ch. II, Theorem 5.4]).

An important consequence of the local cross-section theorem is the following:

Lemma 2.3. *Let G be a compact Lie group, X a free G-space. Then the orbit projection $p : X \to X/G$ is a locally trivial fibration.*

Before proceeding with the proof of the lemma we recall that a locally trivial fibration here means that every point $p(x) \in X/G$ admits a neighborhood U such that the inverse image $p^{-1}(U)$ is G-equivalent to $G \times U$, i.e., there exists a G-homeomorphism $f : p^{-1}(U) \to G \times U$ such that the following diagram commutes: