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Group-valued continuous functions with the topology of pointwise convergence

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ABSTRACT

Let G be a topological group with the identity element e. Given a space X, we denote by $C_p(X,G)$ the group of all continuous functions from X to G endowed with the topology of pointwise convergence, and we say that X is: (a) G-*regular* if, for each closed set $F \subseteq X$ and every point $x \in X \setminus F$, there exist $f \in C_p(X,G)$ and $g \in G \setminus \{e\}$ such that $f(x) = g$ and $f(F) \subseteq \{e\}$; (b) $G^\star$-*regular* provided that there exists $g \in G \setminus \{e\}$ such that, for each closed set $F \subseteq X$ and every point $x \in X \setminus F$, one can find $f \in C_p(X,G)$ with $f(x) = g$ and $f(F) \subseteq \{e\}$. Spaces X and Y are G-*equivalent* provided that the topological groups $C_p(X,G)$ and $C_p(Y,G)$ are topologically isomorphic.

We investigate which topological properties are preserved by G-equivalence, with a special emphasis being placed on characterizing topological properties of X in terms of those of $C_p(X,G)$. Since $\mathbb{R}$-equivalence coincides with l-equivalence, this line of research "includes" major topics of the classical C_p-theory of Arhangel'skiĭ as a particular case (when $G = \mathbb{R}$).

We introduce a new class of TAP groups that contains all groups having no small subgroups (NSS groups). We prove that: (i) for a given NSS group G, a G-regular space X is pseudocompact if and only if $C_p(X,G)$ is TAP, and (ii) for a metrizable NSS group G, a $G^\star$-regular space X is compact if and only if $C_p(X,G)$ is a TAP group of countable tightness. In particular, a Tychonoff space X is pseudocompact (compact) if and only if $C_p(X,\mathbb{R})$ is a TAP group (of countable tightness). Demonstrating the limits of the result in (i), we give an example of a precompact TAP group G and a G-regular countably compact space X such that $C_p(X,G)$ is not TAP.

We show that Tychonoff spaces X and Y are $\mathbb{T}$-equivalent if and only if their free precompact Abelian groups are topologically isomorphic, where $\mathbb{T}$ stays for the quotient group $\mathbb{R}/\mathbb{Z}$. As a corollary, we obtain that $\mathbb{T}$-equivalence implies G-equivalence for every Abelian precompact group G. We establish that $\mathbb{T}$-equivalence preserves the following topological properties: compactness, pseudocompactness, σ-compactness, the property of being a Lindelöf Σ-space, the property of being a compact metrizable space, the (finite) number of connected components, connectedness, total disconnectedness. An example of $\mathbb{R}$-equivalent (that is, l-equivalent) spaces that are not $\mathbb{T}$-equivalent is constructed.

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In notation and terminology we follow [6] and [9] if not stated otherwise. *All topological spaces are assumed to be Tychonoff* (*that is, completely regular T_1 spaces*) *and nonempty, and all topological groups are assumed to be Hausdorff.*

By $\mathbb{N}$ we denote the set of all natural numbers, ω stays for the least nonzero limit ordinal, $\mathbb{Z}$ is the discrete additive group of integers, $\mathbb{R}$ is the additive group of reals with its usual topology, $\mathbb{T}$ stays for the quotient group $\mathbb{R}/\mathbb{Z}$, and $\mathbb{Z}(n)$ denotes the cyclic group of order n (with the discrete topology). The identity element of a group G is denoted by e_G, or simply by e when there is no danger of confusion.

If G is a topological group, then the symbol $\widehat{G}$ stays for the completion of G with respect to the two-sided uniformity. If $G = \widehat{G}$, then G is called *complete*. It is well known that $\widehat{G}$ always exists, $\widehat{G}$ is a topological group, G is dense in $\widehat{G}$, and if G is a dense subgroup of a complete group H, then $\widehat{G} = H$. If G is a subgroup of some compact group, then G is called *precompact*.

1. Introduction

Definition 1.1. Let X be a space and G a topological group.

(i) We shall use $C(X,G)$ to denote the group of all continuous functions from X to G, equipped with the "pointwise group operations". That is, the product of $f \in C(X,G)$ and $g \in C(X,G)$ is the function $fg \in C(X,G)$ defined by $fg(x) = f(x)g(x)$ for all $x \in X$, and the inverse element of f is the function $h \in C(X,G)$ defined by $h(x) = (f(x))^{-1}$ for all $x \in X$.

(ii) The family

$$\{W(x,U):\ x \in X,\ U \text{ is an open subset of } G\},$$

where

$$W(x,U) = \{f \in C(X,G):\ f(x) \in U\},$$

forms a subbase of the *topology of pointwise convergence* on $C(X,G)$. We use the symbol $C_p(X,G)$ to denote the set $C(X,G)$ endowed with this topology.

One can easily see that $C_p(X,G)$ is a topological group.

Definition 1.2. Let G and H be topological groups.

(i) Recall that G and H are said to be *topologically isomorphic* if there exists a bijection $f: G \to H$ which is both a group homomorphism and a homeomorphism. We write $G \cong H$ whenever G and H are topologically isomorphic.

(ii) We say that spaces X and Y are *G-equivalent*, and denote this by $X \overset{G}{\sim} Y$, provided that $C_p(X,G) \cong C_p(Y,G)$.

(iii) Let $\mathscr{C}$ be a class of spaces. We say that a topological property $\mathscr{E}$ is *preserved by G-equivalence within the class $\mathscr{C}$* provided that the following condition holds: If $X \in \mathscr{C}$, $Y \in \mathscr{C}$, $X \overset{G}{\sim} Y$ and X has the property $\mathscr{E}$, then Y must have the property $\mathscr{E}$ as well. The sentence "$\mathscr{E}$ is *preserved by G-equivalence*" is used as an abbreviation for "$\mathscr{E}$ is preserved by G-equivalence within the class of Tychonoff spaces".

(iv) Given a class $\mathscr{C}$ of spaces, we say that *G-equivalence implies H-equivalence within the class $\mathscr{C}$* provided that the following statement holds: If $X \in \mathscr{C}$, $Y \in \mathscr{C}$ and $X \overset{G}{\sim} Y$, then $X \overset{H}{\sim} Y$. The sentence "*G-equivalence implies H-equivalence*" shall be used as an abbreviation for "G-equivalence implies H-equivalence within the class of Tychonoff spaces".

In [15] Markov has introduced the free topological group $F(X)$ of a space X and defined spaces X and Y to be *M-equivalent* if $F(X) \cong F(Y)$. Thereafter, a significant effort went into an investigation of how topological properties of $F(X)$ depend on those of X, as well as which topological properties are preserved by M-equivalence.

Every continuous function $f: X \to G$ from a space X to a topological group G can be (uniquely) extended to a continuous group homomorphism $\widehat{f}: F(X) \to G$. This elementary fact (with $\mathbb{T}$ as G) was applied by Graev to show that the closed unit interval and the circle are not M-equivalent [10]. Tkachuk noticed in [22] that M-equivalence implies G-equivalence for every Abelian topological group G. He then applied this observation to $G = \mathbb{Z}(2)$ to show that connectedness is preserved by M-equivalence [22].

Later on, many properties of M-equivalence were discovered by means of the notion of l-equivalence; see [1]. Recall that spaces X and Y are called *l-equivalent* provided that $C_p(X,\mathbb{R})$ and $C_p(Y,\mathbb{R})$ are *topologically isomorphic as topological vector spaces*. A fundamental observation pertinent to the subject of this paper has been made in [22] by Tkachuk: spaces X and Y are l-equivalent if and only if $C_p(X,\mathbb{R})$ and $C_p(Y,\mathbb{R})$ are *topologically isomorphic as topological groups*. In other words, l-equivalence of spaces coincides with their $\mathbb{R}$-equivalence (in our notation). A far reaching conclusion that one might get from this fact is that, despite a significant emphasis on the *topological vector space* structure commonly placed in the C_p-theory [1], this structure is largely irrelevant to the study of the notion of l-equivalence, and in fact may as well be replaced by the *topological group* structure. It is this conclusion that led us to an idea of introducing the general notion of G-equivalence, for an arbitrary topological group G.

This opens up a topic of studying the properties of the topological group $C_p(X,G)$, for a given space X and a topological group G. Let us outline major problems that appear to be of particular interest in this new area of research.