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Degree bounds—An invitation to postmodern invariant theory

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In memory of C. and C.

Abstract

This is an invitation to invariant theory of finite groups; a field where methods and results from a wide range of mathematics merge to form a new exciting blend. We use the particular problem of finding degree bounds to illustrate this.

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Invariant theory is a relatively young field. Although we will give explicit definitions later, a brief discussion on its short but illustrious history can give insight into this area of mathematics:

In 1773, J.L. Lagrange observed that the determinant of a binary quadratic form is an invariant polynomial under any linear transformation.

Around 1800, Carl F. Gauss considered the general problem of the invariance of binary quadratic forms with integral coefficients under an action of the special linear group. However, it was George Boole who, in 1843, laid the foundations of invariant theory with his papers [5,6]. In the second half of the 19th century, invariant theory blossomed under the hands of people like Sir Arthur Cayley in England and James J. Sylvester in England and the US, Felix Klein in Germany, and many others. Indeed, the history of invariant theory in the 19th century is exciting and rich, and we recommend the survey articles by W. Franz Meyer [33,34], and of course the lovely lectures by Felix Klein [29].

A culmination point of this early phase was Paul Gordan's proof that the invariant binary forms (under an action of the special linear group $SL(2, \mathbb{C})$) are finitely generated (see [22]). However, the early history of invariant theory ended abruptly with David Hilbert's proof of the finite generatedness of the $SL(n, \mathbb{C})$-invariants using, or better creating, completely new tools and paradigms (see [24]). Indeed, David Hilbert's viewpoint was earthshaking and rang in mathematical modernity.[2] However, the school around David Hilbert and later Emmy Noether was just too successful; they solved the current problems of invariant theory of finite groups, leaving behind a seemingly complete, and thus dead branch of mathematics (see [15] and the response [46]).

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[2] And, in particular, modern algebra.

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On the other hand, modular invariant theory over finite fields was a field presenting problems and questions of a completely different type. These were studied by Leonard E. Dickson and his school at Chicago. Impressive as these results are (Leonard E. Dickson's collected works, 6 huge volumes [13], are a treasure box for everybody who works in the area) the tools available back then were soon exhausted, and so no wonder that also his algorithmic approach faded.

However, modernity that had marginalized classical invariant theory at the same time brought new questions and problems to invariant theory. In particular, the entire field of modular invariant theory over finite fields was revitalized by algebraic topology.[3] Nowadays there are many other connections and overlaps with fields like group cohomology,[4] representation theory, algebraic combinatorics (see [49] for an idiosyncratic look back), commutative algebra, and number theory.

Also, the electronic revolution of the 20th century and in the course of that the development of powerful computer algebra programs allows to tackle problems that were out of reach for, e.g., Dickson's school. And, of course, vice versa: hard algorithmic problems in invariant theory serve as motivation for the development of such software.

But not only does invariant theory profit from the neighbouring areas; these areas also take advantage of its results. And indeed, invariant-theoretic problems have been discovered in almost every area of mathematics. Moreover, invariant theory is continuously challenged by daily life problems in physics (think of Emmy Noether's milestone paper [43]), coding theory (see the survey article [54] for more information on this), numerical analysis (see, e.g., [61]), engineering (e.g., in the production of rubbermats for the car industry [32]), and elsewhere (see Chapter 5 of [11] for an overview).

1. Introduction

Consider a quadratic polynomial

$$p(x, y) = ax^2 + 2bxy + cy^2 \in \mathbb{C}[x, y]$$

in two variables x, y with complex coefficients, i.e., a *binary quadratic form*. If we replace the variables x, y by

$$x' = x + y, \quad \text{and} \quad y' = y,$$

we obtain the polynomial

$$p'(x, y) = p(x + y, y) = a'x^2 + 2b'xy + c'y^2,$$

where

$$a' = a, \qquad b' = b + a, \quad \text{and} \quad c' = c + 2b + a.$$

We observe that the coefficients of both polynomials satisfy the following equation

$$ac - b^2 = a'c' - b'^2.$$

This in not an accident. Indeed, let $p(x, y)$ be an arbitrary binary quadratic form, and let T be a linear transformation given by

$$\mathsf{T} \quad \begin{cases} x \mapsto x + \lambda y, & \lambda \in \mathbb{C}, \\ y \mapsto y. \end{cases}$$

Then the *determinant* of p, $\det(p) = ac - b^2$, remains unchanged under the T-action:

$$\det(p) = \det(p'),$$

where $p'(x, y) = p(x + \lambda y, y)$. This observation was made by Joseph Louis Lagrange in 1773 (see [31, Corollaire II]).

Let us rephrase this result in order to put it into a broader context.

[3] We recover our rings of invariants as rings of invariants of Weyl groups acting on the cohomology ring of elementary Abelian p-groups, if we ignore the exterior part in odd characteristic.

[4] The ring of invariants is just the zeroth cohomology of the group in question with twisted coefficients in $\mathbb{F}[V]$, see, e.g., [1].