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Productive local properties of function spaces

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Abstract

We characterize the spaces X for which the space $C_p(X)$ of real valued continuous functions with the topology of pointwise convergence has local properties related to the preservation of countable tightness or the Fréchet property in products. In particular, we use the methods developed to construct an uncountable subset W of the real line such that the product of $C_p(W)$ with any strongly Fréchet space is Fréchet. The example resolves an open question.

MSC: 54C30; 54C35; 54D55; 54G10

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1. Introduction

We study the relationship between a space X and the function space $C_p(X)$ of real-valued continuous functions on X with the topology of pointwise convergence. The space $C_p(X)$ has been extensively studied [18]. It has been the source of many interesting examples and often provides information about its underlying space.

In particular, we will be interested in when $C_p(X)$ has local properties related to the preservation of certain local properties in products. Conditions on X that characterize the properties of being Fréchet or countably tight (for definitions see the next section) and other relatively simple local convergence properties of $C_p(X)$ have long been known (see [8,1]). The properties of countable tightness and Fréchet are not productive (see [22,2]). In recent years local properties have been defined that allow for some product results to hold. We characterize the spaces X such that $C_p(X)$ is productively Fréchet, productively countably tight, and tight.

These characterizations allow us to provide an example of a space answering a question of Gruenhage. In particular, Gruenhage has asked [11] whether every productively Fréchet space has a regular countably compact extension that is also Fréchet. Under CH we give a negative answer to this question. In fact, we construct an uncountable $X \subseteq \mathbb{R}$ such that $C_p(X)$ is productively Fréchet but not tight. This example gives a nonmetrizable separable productively Fréchet topological group which has no regular countably compact extension.

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2. Terminology and background results

We use standard set theory notation. Ordinals are identified with their set of predecessors. By ω and ω_1 we denote the first infinite ordinal and first uncountable ordinal respectively. For a set X we denote the finite, countable, and countably infinite subsets of X by $[X]^{<\omega}$, $[X]^{\leqslant\omega}$, and $[X]^{\omega}$, respectively.

Let X be a completely regular topological space. We let $C_p(X)$ denote the set of all real-valued continuous functions defined on X with the topology of pointwise convergence. By $\bar{0}$ we denote the function which has value 0 at each point in X. For each $F \in [X]^{<\omega}$ and $k \in \omega$ let $U(F,k)$ be the set of all $f \in C_p(X)$ such that $f[F] \subseteq (-1/2^k, 1/2^k)$. Notice that the collection $\mathcal{N}(\bar{0}) = \{U(F,k)\colon F \in [X]^{<\omega} \text{ and } k \in \omega\}$ is a base of open neighborhoods at $\bar{0}$. For more information on $C_p(X)$ see [18].

It will be convenient for us to use the language of filters. Let X be a fixed set. Recall that a collection $\mathcal{F}$ of nonempty subsets of X is a filter provided that $F \cap G \in \mathcal{F}$ for all $G, F \in \mathcal{F}$ and for any $F \in \mathcal{F}$ and H such that $F \subseteq H \subseteq X$ we have $H \in \mathcal{F}$. Given a collection $\mathcal{C}$ of sets we let $\mathcal{C}^{\uparrow} = \{S\colon \exists C \in \mathcal{C},\ C \subseteq S \subseteq X\}$. Clearly, if $\mathcal{C}$ is closed under finite intersections and does not contain the empty set, then $\mathcal{C}^{\uparrow}$ is a filter. If a filter is of the form $\mathcal{F} = \mathcal{C}^{\uparrow}$ we say that $\mathcal{C}$ is a base for $\mathcal{F}$. Generally, if $\mathcal{C}$ is a base for $\mathcal{F}$ we write $\mathcal{F} = \mathcal{C}$ as opposed to $\mathcal{F} = C^{\uparrow}$ when there is no danger of confusion. A sequence $(x_n)_{n\in\omega}$ is identified with the filter $\{\{x_k\colon k \geqslant n\}\colon n \in \omega\}^{\uparrow}$ generated by its tails.

We say two filters $\mathcal{F}$ and $\mathcal{G}$ *mesh*, in symbols $\mathcal{F}\#\mathcal{G}$, provided that for every $F \in \mathcal{F}$ and $G \in \mathcal{G}$, $F \cap G \neq \emptyset$. Given two filters that mesh, the *supremum* of $\mathcal{F}$ and $\mathcal{G}$ is defined by $\mathcal{F} \vee \mathcal{G} = \{F \cap G\colon F \in \mathcal{F} \text{ and } G \in \mathcal{G}\}^{\uparrow}$. Given two filters $\mathcal{F}$ and $\mathcal{G}$, we say that $\mathcal{F}$ is *finer* than $\mathcal{G}$, written $\mathcal{G} \leqslant \mathcal{F}$, provided that for every $G \in \mathcal{G}$ there is an $F \in \mathcal{F}$ such that $F \subseteq G$. Given a collection of filters $\{\mathcal{G}_\alpha\colon \alpha \in I\}$ we define the *infimum* of the collection to be the filter $\bigwedge_{\alpha\in I}\mathcal{G}_\alpha$ generated by the collection $\{\bigcup_{\alpha\in I} f(\alpha)\colon f \in \prod_{\alpha\in\mathcal{I}}\mathcal{G}_\alpha\}$. In particular, a filter $\mathcal{F}\#(\bigwedge_{\alpha\in I}\mathcal{G}_\alpha)$ if and only if for every $F \in \mathcal{F}$ there is an $\alpha \in I$ such that $F\#\mathcal{G}_\alpha$.

Given $R \subseteq X \times Y$ we denote its inverse by $R^{-1} = \{(y,x)\colon (x,y) \in R\}$. If $F \subseteq X$ and $R \subseteq X \times Y$ we let $RF = \{y\colon (x,y) \in R \text{ and } x \in F\}$. If $\mathcal{F}$ is a filter on X we let $R\mathcal{F} = \{RF\colon F \in \mathcal{F}\}^{\uparrow}$ provided that $RF \neq \emptyset$ for all $F \in \mathcal{F}$. We say a class $\mathbb{D}$ of filters is $\mathbb{F}_1$-*composable* provided that for any sets X and Y and any set $R \subseteq X \times Y$ if $\mathcal{F}$ is a $\mathbb{D}$ filter on X and $R\mathcal{F}$ is defined, then $R\mathcal{F}$ is a $\mathbb{D}$-filter on Y. Notice that if $\mathbb{K}$ is a $\mathbb{F}_1$-composable class of filters, $f\colon X \to Y$ is a function, $\mathcal{F}$ is $\mathbb{K}$-filter on X, and $\mathcal{G}$ is $\mathbb{K}$-filter meshing with $f[Y]$; then $f[\mathcal{F}] = \{f[F]\colon F \in \mathcal{F}\}^{\uparrow}$ and $f^{-1}[\mathcal{G}] = \{f^{-1}(G)\colon G \in \mathcal{G}\}^{\uparrow}$ are both $\mathbb{K}$-filters. A useful observation is that if $\mathcal{F}$ is a filter on X, $\mathcal{G}$ is a filter on Y, and $R \subseteq X \times Y$; then $R\mathcal{F}\#\mathcal{G}$ if and only if $R^{-1}\mathcal{G}\#\mathcal{F}$.

If $\mathcal{F}$ has a base consisting of one (countably many) set(s), then we say $\mathcal{F}$ is *principal* (*countably based*). We say $\mathcal{F}$ is *Fréchet* (*strongly Fréchet* [16]) provided that for every principal (countably based) filter $\mathcal{H}$, $\mathcal{H}\#\mathcal{F}$ implies that there is a countably based filter $\mathcal{C}$ such that $\mathcal{C} \geqslant \mathcal{F} \vee \mathcal{H}$. The strongly Fréchet filters are also commonly known as *Fréchet*-α_4 or *countably bisequential* filters. We say $\mathcal{F}$ is *productively Fréchet* (see [9,13]) provided that for every strongly Fréchet filter $\mathcal{H}$ if $\mathcal{H}\#\mathcal{F}$, then there is a countably based filter $\mathcal{C}$ such that $\mathcal{C} \geqslant \mathcal{F} \vee \mathcal{H}$. We say $\mathcal{F}$ is *bisequential* [17] provided for that every filter $\mathcal{G}$ such that $\mathcal{G}\#\mathcal{F}$ there is a countably based filter $\mathcal{L}$ such that $\mathcal{L} \geqslant \mathcal{F}$ and $\mathcal{L}\#\mathcal{G}$.

We say $\mathcal{F}$ is *countably tight* [2] if for any set C such that $C\#\mathcal{F}$ there is a countable $D \subseteq C$ such that $D\#\mathcal{F}$. We say $\mathcal{F}$ is *tight* (see [14,5]) provided that for any collection $\{(x_n^\alpha)_{n\in\omega}\colon \alpha \in I\}$ of sequences such that $(\bigwedge_{\alpha\in I}(x_n^\alpha)_{n\in\omega})\#\mathcal{F}$, there is a countable $J \subseteq I$ such that $(\bigwedge_{\alpha\in J}(x_n^\alpha)_{n\in\omega})\#\mathcal{F}$. The definition of tightness we give here is from [14] and is shown there to be equivalent to the original definition in [4]. We say $\mathcal{F}$ is *productively countably tight* provided that for any collection $\{\mathcal{G}_\alpha\colon \alpha \in I\}$ of countably tight filters such that $(\bigwedge_{\alpha\in I}\mathcal{G}_\alpha)\#\mathcal{F}$, there is a countable $J \subseteq I$ such that $(\bigwedge_{\alpha\in J}\mathcal{G}_\alpha)\#\mathcal{F}$. The definition of productive countable tightness we give here is from [14] and is shown there to be equivalent to a class of filters defined in [2] for completely regular spaces.

For each of the types of filters mentioned in the preceding two paragraphs we say a space X is of the same type if all of its neighborhood filters are of that type. For example, we say X is a strongly Fréchet space provided that all of its neighborhood filters are strongly Fréchet.

In [2] a completely regular space X is called $\aleph_0$-*bisequential* provided that every countable subset of X is bisequential and ω is in the frequency spectrum of X. By [2, Theorem 3.6] and [14, Proposition 43] the statement that ω is in the frequency spectrum of X is equivalent to saying that X is productively countably tight for completely regular X.

It is known that a topological space is productively Fréchet if and only if its product with every strongly Fréchet space is Fréchet (see [9,13]). By [14, Corollary 39], a space is tight if and only if its product with every Fréchet space is countably tight. By [14, Corollary 36] and [2, Theorem 3.6], a space is productively countably tight if and only if