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An alternative definition of coarse structures

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Abstract

Roe [J. Roe, Lectures on Coarse Geometry, University Lecture Series, vol. 31, Amer. Math. Soc., Providence, RI, 2003] introduced coarse structures for arbitrary sets X by considering subsets of $X \times X$. In this paper we introduce large scale structures on X via the notion of uniformly bounded families and we show their equivalence to coarse structures on X. That way all basic concepts of large scale geometry (asymptotic dimension, slowly oscillating functions, Higson compactification) have natural definitions and basic results from metric geometry carry over to coarse geometry.

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1. Introduction

Recall that the *star* $\mathrm{St}(B,\mathcal{U})$ of a subset B of X with respect to a family $\mathcal{U}$ of subsets of X is the union of those elements of $\mathcal{U}$ that intersect B. Given two families $\mathcal{B}$ and $\mathcal{U}$ of subsets of X, $\mathrm{St}(\mathcal{B},\mathcal{U})$ is the family $\{\mathrm{St}(B,\mathcal{U})\}$, $B \in \mathcal{B}$, of all stars of elements of $\mathcal{B}$ with respect to $\mathcal{U}$.

Definition 1.1. A *large scale structure* $\mathcal{LSS}_X$ on a set X is a non-empty set of families $\mathcal{B}$ of subsets of X (called *uniformly bounded* or *uniformly* $\mathcal{LSS}_X$*-bounded* once $\mathcal{LSS}_X$ is fixed) satisfying the following conditions:

(1) $\mathcal{B}_1 \in \mathcal{LSS}_X$ implies $\mathcal{B}_2 \in \mathcal{LSS}_X$ if each element of $\mathcal{B}_2$ consisting of more than one point is contained in some element of $\mathcal{B}_1$.
(2) $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{LSS}_X$ implies $\mathrm{St}(\mathcal{B}_1, \mathcal{B}_2) \in \mathcal{LSS}_X$.

We think of (2) above as a generalization of the triangle inequality.

The *trivial extension* $e(\mathcal{B})$ of a family $\mathcal{B}$ is defined as $\mathcal{B} \cup \{\{x\}\}_{x\in X}$. Recall that $\mathcal{B}$ is a *refinement* of $\mathcal{B}'$ if every element of $\mathcal{B}$ is contained in some element of $\mathcal{B}'$. Thus, the meaning of (1) of Definition 1.1 is that if $\mathcal{B} \in \mathcal{LSS}_X$, then all refinements of $e(\mathcal{B})$ also belong to $\mathcal{LSS}_X$.

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Proposition 1.2. *Any large scale structure $\mathcal{LSS}_X$ on X has the following properties*:

(1) *$\mathcal{B} \in \mathcal{LSS}_X$ if each element of $\mathcal{B}$ consists of at most one point.*
(2) *$\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{LSS}_X$ implies $\mathcal{B}_1 \cup \mathcal{B}_2 \in \mathcal{LSS}_X$.*

Proof. (1) Pick any $\mathcal{B}_1 \in \mathcal{LSS}_X$ and notice $\mathcal{B}_2 := \mathcal{B}$ satisfies (1) of Definition 1.1.

(2) Let $\mathcal{B}'_i := e(\mathcal{B}_i)$ for $i = 1, 2$. Observe $\mathcal{B}'_i \in \mathcal{LSS}_X$. Therefore $\mathcal{B}_3 = \mathrm{St}(\mathcal{B}'_1, \mathcal{B}'_2) \in \mathcal{LSS}_X$ and notice any element of $\mathcal{B}_1 \cup \mathcal{B}_2$ is contained in an element of $\mathcal{B}_3$. □

We have two basic examples of large scale structures induced by other structures on X. The first one deals with metric spaces, so let us point out there is no need to restrict ourselves to metrics assuming only finite values. To emphasize that, let us call $d : X \times X \to R_+ \cup \infty$ an *∞-metric* if it satisfies all the regular axioms of a metric (with the understanding that $x + \infty = \infty$). Notice that ∞-metrics have the advantage over regular metrics in the fact that one can easily define the *disjoint union* $\bigoplus_{s\in S}(X_s, d_s)$ of any family of ∞-metric spaces (X_s, d_s). Namely, put $d(x, y) = \infty$ if x and y belong to different spaces X_s and X_t (those are assumed to be disjoint). Conversely, any ∞-metric space (X, d) is the disjoint union of its finite components $(C, d|C)$ (two elements belong to the same finite component if $d(x, y) < \infty$).

Proposition 1.3. *Any ∞-metric space (X, d) has a natural large scale structure $\mathcal{LSS}(X, d)$ defined as follows*:

$\mathcal{B} \in \mathcal{LSS}(X, d)$ if and only if there is $M > 0$ such that all elements of $\mathcal{B}$ are of diameter at most M.

Proof. If $\mathcal{B}_1 \in \mathcal{LSS}(X, d)$ and for each $B_\beta \in \mathcal{B}_2$ consisting of more than one point there is a $B_\alpha \in \mathcal{B}_1$ containing B_β, then $\mathrm{diam}(B_\beta) \leqslant \mathrm{diam}(B_\alpha) \leqslant M$ for each $B_\beta \in \mathcal{B}_2$, whence $\mathcal{B}_2 \in \mathcal{LSS}(X, d)$. If $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{LSS}(X, d)$ then there are $M_1, M_2 > 0$ such that $\mathrm{diam}(B_\alpha) \leqslant M_1$ and $\mathrm{diam}(B_\beta) \leqslant M_2$ for all $B_\alpha \in \mathcal{B}_1$, $B_\beta \in \mathcal{B}_2$. Thus for any $B_\alpha \in \mathcal{B}_1$, $\mathrm{diam}(\mathrm{St}(B_\alpha, \mathcal{B}_2)) \leqslant 2M_2 + M_1$, whence $\mathrm{St}(\mathcal{B}_1, \mathcal{B}_2) \in \mathcal{LSS}(X, d)$. It follows that $\mathcal{LSS}(X, d)$ is a large scale structure. □

One can generalize Proposition 1.3 as follows: Given certain families $\mathcal{F}$ of positive functions from an ∞-metric space X to reals one can define $\mathcal{LSS}(X, \mathcal{F})$ by declaring $\mathcal{B} \in \mathcal{LSS}(X, \mathcal{F})$ if and only if there is $f \in \mathcal{F}$ such that $\mathcal{B}$ refines the family of balls $\{B(x, f(x))\}_{x\in X}$.

One family of interest is all f such that $\lim_{x\to\infty} \frac{f(x)}{d(x,x_0)} = 0$, where x_0 is a fixed point in a metric space X (if X is an ∞-metric space, one needs to look at each finite component separately). That leads to the *sublinear large scale structure* on X introduced by Dranishnikov and Smith [5] (see also [4]).

Proposition 1.4. *Any group $(X, \cdot)$ has a natural large scale structure $\mathcal{LSS}_l(X, \cdot)$ defined as follows*:

$\mathcal{B} \in \mathcal{LSS}_l(X, \cdot)$ if and only if there is a finite subset F of X such that $\mathcal{B}$ refines the shifts $\{x \cdot F\}_{x\in X}$ of F.

Proof. Notice that if $\mathcal{B} \neq \emptyset$ refines $\{x \cdot F\}_{x\in X}$ for some finite subset F of X, then $e(\mathcal{B})$ also refines $\{x \cdot F\}_{x\in X}$.

Suppose $\mathcal{B}_i$ refines $\{x \cdot F_i\}_{x\in X}$ for $i = 1, 2$, where F_1 and F_2 are finite subsets of X. We may enlarge F_2 and assume it is symmetric ($y \in F_2$ implies $y^{-1} \in F_2$).

Let F be the set of all products $x \cdot y \cdot z$, where $x \in F_1$ and $y, z \in F_2$. Given $B \in \mathcal{B}_1$ pick $a \in X$ such that $B \subset a \cdot F_1$. If $B' \in \mathcal{B}_2$ and $u \in B \cap B'$, choose $y \in X$ so that $B' \subset y \cdot F_2$. Thus $u = a \cdot f_1 = y \cdot f_2$, where $f_1 \in F_1$ and $f_2 \in F_2$. Therefore $y = a \cdot f_1 \cdot f_2^{-1}$ and $B' \subset a \cdot F$ proving that $\mathrm{St}(B, \mathcal{B}_2) \subset a \cdot F$. □

Remark 1.5. Notice that any group $(X, \cdot)$ has another natural large scale structure $\mathcal{LSS}_r(X, \cdot)$ defined as follows:

$\mathcal{B} \in \mathcal{LSS}_r(X, \cdot)$ if and only if there is a finite subset F of X such that $\mathcal{B}$ refines the shifts $\{F \cdot x\}_{x\in X}$ of F.

Clearly, the two structures coincide if X is Abelian. However, they may differ even for finitely presented virtually Abelian groups.