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Characterization of compact spaces with noncoinciding dimensions which are subsets of products of simple spaces ☆

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ABSTRACT

In this paper we characterize closed subsets of products of simple compact spaces with noncoinciding dimensions dim and ind.

1. Introduction

In 1935 P.S. Alexandroff raized the question about the relationships between three main dimensions dim, ind, and Ind in the class of compacta and later showed that $\dim X \leqslant \operatorname{ind} X$ for any compactum X. First examples of compacta with noncoinciding dimensions dim and ind were constructed by A. Lunc and V. Lokucievskiĭ in 1949. Then more examples of compacta with additional besides dimensional properties appeared.

In [8] B. Pasynkov introduced the notion of a tailing of a space and proposed methods for constructing compacta with noncoinciding dimensions dim and ind which were developed in [4,5]. It turned out that a lot of compacta with noncoinciding dimensions dim and ind are realized as tailings. Besides, compacta constructed by A. Lunc, S. Mardešić, B. Pasynkov, P. Vopěnka are subsets of the topological products of simple spaces. Thus the following problem was stated by B. Pasynkov: Characterize compact subsets of topological products with noncoinciding dimensions. It is also worth noting that closely connected questions about dimensions of subsets of products were considered in [6,7]. In this work a partial answer on this question is given.

Below a space means a topological space. A compactum is a Hausdorff compact space. A map—a continuous mapping between spaces. The abbreviation for neighbourhood(s) is nbd(s), cf. for ordinal denotes it cofinality, $I = [0, 1]$. By cl_X, int_X, bd_X we denote closure, interior, and boundary of the set in the space X respectively.

All information about, dimensions may be found in [1] or [3] and we follow the notations from [2].

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2. Preliminaries

Pair of nonempty open in X disjoint (briefly odp) sets O_i, $i = 1, 2$, defines a partition $F = X \setminus (O_1 \cup O_2)$ in X. The odp O_i, $i = 1, 2$, is *essential* if $\mathrm{cl}_X O_1 \cap \mathrm{cl}_X O_2 \neq \emptyset$. If $O_1 \cup O_2$ is dense in X then O_i, $i = 1, 2$, is called oddp in X [5]. For an odp O_i, $i = 1, 2$, in X $\mathrm{cl}_X O_1 \cap \mathrm{cl}_X O_2 = \mathrm{bd}_X O_1 \cap \mathrm{bd}_X O_2$,

Lemma 2.1. *Let O_i, $i = 1, 2$, be an unessential odp in a normal space X, and for partition F defined by it* $\mathrm{Ind}\, F \leqslant n$, $n \in \{-1, 0\} \cup \mathbb{N}$. *Then there exists odp U_i, $i = 1, 2$, with partition F' defined by it, such that* $\mathrm{cl}_X O_i \subset U_i$, $i = 1, 2$, *and* $\mathrm{Ind}\, F' \leqslant \max\{-1, n-1\}$.

Proof. The sets $\mathrm{cl}_X O_i \cap F$, $i = 1, 2$, are closed disjoint subsets of F. If both of them are not empty then since $\mathrm{Ind}\, F \leqslant n$, there exists odp O'_i, $i = 1, 2$, in F with partition F' defined by it such that $\mathrm{cl}_X O_i \cap F \subset O'_i$, $i = 1, 2$, and $\mathrm{Ind}\, F' \leqslant n - 1$. It is easy to check that the sets $U_i = \mathrm{cl}_X O_i \cup O'_i$, $i = 1, 2$, are an odp in X which defines partition F'.

If, for example, $\mathrm{cl}_X O_1 \cap F = \emptyset$ then the sets $U_1 = \mathrm{cl}_X O_1$ and $U_2 = F \cup \mathrm{cl}_X O_2$ are the required disjoint clopen pair in X and the partition defined by it is empty. □

Corollary 2.2. *Let X be a normal space connected between points x_1 and x_2. Then any odp O_i, $i = 1, 2$, such that $x_i \in O_i$, $i = 1, 2$, and the partition defined by it is strongly zero-dimensional is essential.*

Lemma 2.3. *Let X be a hereditarily normal space, O_i, $i = 1, 2$—odp in X and F—a partition in X defined by it. If* $\mathrm{Ind}(\mathrm{cl}_X O_1 \cap \mathrm{cl}_X O_2) \leqslant m$ *and* $\mathrm{Ind}(F \setminus (\mathrm{cl}_X O_1 \cap \mathrm{cl}_X O_2)) \leqslant n$, $n, m \in \{-1, 0\} \cup \mathbb{N}$, *then there exists odp U_i, $i = 1, 2$, with partition F' defined by it such that $O_i \subset U_i$, $i = 1, 2$, and* $\mathrm{Ind}\, F' \leqslant \max\{m, n-1\}$.

Proof. Put $X' = X \setminus (\mathrm{cl}_X O_1 \cap \mathrm{cl}_X O_2)$ and $O'_i = X' \cap O_i$, $i = 1, 2$. Since the odp O'_i, $i = 1, 2$, is unessential in a normal space X' and for the partition $T = X' \cap F$ defined by it $\mathrm{Ind}\, T \leqslant n$ there exists by Lemma 2.1 odp U_i, $i = 1, 2$, and a partition T' in X' defined by it, such that $\mathrm{cl}_X O_i \cap X' \subset U_i$, $i = 1, 2$, and $\mathrm{Ind}\, T' \leqslant n - 1$. Since U_i, $i = 1, 2$, is also an odp in X so $X \setminus (U_1 \cup U_2) = F' = (\mathrm{cl}_X O_1 \cap \mathrm{cl}_X O_2) \cup T'$. The set $\mathrm{cl}_X O_1 \cap \mathrm{cl}_X O_2$ is closed in F', $\mathrm{Ind}(\mathrm{cl}_X O_1 \cap \mathrm{cl}_X O_2) \leqslant m$, $F' \setminus (\mathrm{cl}_X O_1 \cap \mathrm{cl}_X O_2) = T'$ and $\mathrm{Ind}\, T' \leqslant n - 1$. From Dowker's theorem (see, for example, [1, Chapter 7, §2, Theorem 2]) it follows that $\mathrm{Ind}\, F' \leqslant \max\{m, n-1\}$. □

Lemma 2.4. *Let $X = X_1 \cup X_2$, where X_i are closed in X, $i = 1, 2$, and X_2 is hereditarily normal. Then*

(i) $\mathrm{ind}\, X \leqslant \max\{\mathrm{ind}\, X_1, \mathrm{Ind}(X_2 \setminus X_1)\}$ *if X is regular*;
(ii) $\mathrm{Ind}\, X \leqslant \max\{\mathrm{Ind}\, X_1, \mathrm{Ind}(X_2 \setminus X_1)\}$ *if X is normal (this is an obvious generalization of Dowker's theorem).*

Proof. (i) Set $\max\{\mathrm{ind}\, X_1, \mathrm{Ind}(X_2 \setminus X_1)\} = n$ and apply induction on $n \geqslant -1$. If $n = -1$ then $X = \emptyset$ and the statement is evident.

Let $n \geqslant 0$. If $x \in X_2 \setminus X_1$ or $x \in X_1 \setminus X_2$ then evidently $\mathrm{ind}_x X \leqslant n$. Put $X_1 \cap X_2 = Y$. Consider $x \in Y$ and a closed set B such that $x \notin B$. If $Y = \{x\}$ then there is clearly a partition C between x and B in X such that $\mathrm{ind}\, C < n$.

If $|Y| > 1$ then we can assume that $B \cap Y \neq \emptyset$. Choose now a partition C_1 between x and $B \cap X_1$ in X_1 such that $\mathrm{ind}\, C_1 < n$. Let also U_i, $i = 1, 2$, be open disjoint subsets of X_1 such that $X_1 \setminus C_1 = U_1 \cup U_2$, $x \in U_1$ and $B \cap X_1 \subset U_2$.

Put $X_2 \setminus C_1 = Z$. Observe that $U_1 \cap Y$ and $(U_2 \cap Y) \cup (B \cap X_2)$ are closed disjoint subsets of the normal space Z. Choose open subsets V_i, $i = 1, 2$, of Z such that $U_1 \cap Y \subset V_1$, $(U_2 \cup Y) \cup (B \cap X_2) \subset V_2$ and $\mathrm{cl}_Z(V_1) \cap \mathrm{cl}_Z(V_2) = \emptyset$.

Note that the sets $A_i = \mathrm{cl}_Z(V_i) \cap (X_2 \setminus X_1)$, $i = 1, 2$, are closed and disjoint in $X_2 \setminus X_1$. Thus there is a partition C_2 between A_1 and A_2 in $X_2 \setminus X_1$ such that $\mathrm{Ind}\, C_2 < n$. Let also O_i, $i = 1, 2$, be open disjoint subsets of $X_2 \setminus X_1$ such that $(X_2 \setminus X_1) \setminus C_2 = O_1 \cup O_2$ and $A_i \subset O_i$, $i = 1, 2$.

Observe that the sets $W_i = V_i \cup O_i$ are open and disjoint in Z, $x \in W_1$ and $B \cap X_2 \subset W_2$. Put $C_2 = X_2 \setminus (W_1 \cup W_2)$. It is evident that $C_2 = (C_1 \cap X_2) \cup C_2$ is a closed subset of X_2, $C_2 \setminus C_1 = C_2$ and the set $C = C_1 \cup C_2$ is a partition between x and B in X. By inductive assumption we have $\mathrm{ind}\, C < n$. The point (i) is proved.

In a similar manner we can prove the point (ii). □

Lemma 2.5. *Let K be a perfectly normal retract of the normal space Y and G_i, $i \in \mathbb{N}$, be the nbd base of K in Y such that* $\mathrm{cl}_Y G_{i+1} \subset G_i$, $i \in \mathbb{N}$, *and $G_1 = Y$. If* $F_i = \mathrm{cl}_Y G_i \setminus G_{i+1}$ *is perfectly normal for $i = 2j$, $j \in \mathbb{N}$, and X is a closed subset of Y then* $\mathrm{Ind}\, X \leqslant \sup\{\mathrm{Ind}(X \cap K), \mathrm{Ind}(X \cap F_i)\colon i \in \mathbb{N}\}$.

Proof. Put $X_i = X \cap F_i$, $i \in \mathbb{N}$, and $X_K = X \cap K$. Set $\sup\{\mathrm{Ind}\, X_K, \mathrm{Ind}\, X_i\colon i \in \mathbb{N}\} = n$, $n \in \mathbb{N} \cup \{-1\}$. Note that the case $n = \infty$ is evident. Apply induction. If $n = -1$ the inequality is obviously true. Let $n \geqslant 0$.

Let A and B be disjoint closed sets in X and put $A_K = K \cap A$, $B_K = K \cap B$. If either $A_K = \emptyset$ or $B_K = \emptyset$ then (suppose $A_K = \emptyset$) there exists $j \in \mathbb{N}$ such that $A \subset X \setminus \mathrm{cl}_Y G_j$. The set $X \setminus G_j$ is a union of closed subsets $\bigcup\{X_i\colon i < j, i \text{ is odd}\}$ and $\bigcup\{X_i\colon i < j, i \text{ is even}\}$, and the second one is a perfectly normal space where both subsets are in fact finite free sums. By