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Universal Borel mappings and Borel actions of groups ☆

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Abstract

In this paper it is proved that for any two saturated (respectively, isometrically ω-saturated) classes (see [S.D. Iliadis, Universal Spaces and Mappings, North-Holland Mathematics Studies, vol. 198, Elsevier, 2005]) $\mathbb{D}$ and $\mathbb{R}$ of separable metrizable (respectively, separable metric) spaces and $\alpha \in \omega^+$ in the class of all Borel mappings of the class α whose domains belong to $\mathbb{D}$ and ranges to $\mathbb{R}$ there exist topologically (respectively, isometrically) universal elements. In particular, $\mathbb{D}$ and $\mathbb{R}$ can be independently one of the following saturated classes of separable metrizable (respectively, separable metric) spaces: (a) the class of all spaces, (b) the class of all countable-dimensional spaces, (c) the class of all strongly countable-dimensional spaces, (d) the class of all locally finite-dimensional spaces, (e) the class of all spaces of dimension less than or equal to a given non-negative integer, and (f) the class of all spaces of dimension ind less than or equal to a given non-finite countable ordinal. This result is not true if instead of the Borel mappings of the class α we shall consider the class of all Borel mappings.

Using the construction of topologically (respectively, isometrically) universal mappings it is proved also that for an arbitrary considered separable metrizable group G and $\alpha \in \omega^+$ in the class of all G-spaces (X, F^X), where X belongs to a given saturated (respectively, isometrically ω-saturated) class $\mathbb{P}$ of spaces and the action F^X of G on X is a Borel mapping of the class α, there exist topologically (respectively, isometrically) universal elements. In particular, $\mathbb{P}$ can be one of the above mentioned saturated (respectively, isometrically ω-saturated) classes of spaces. (About the notions of universality see below.)

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1. Preliminaries

Agreement. All spaces are assumed to be separable metrizable. We shall consider also separable metric spaces. However, this fact will be explicitly verified.

An ordinal is identified with the set of all smaller ordinals. A cardinal is identified with the first ordinal of the corresponding cardinality. By ω we denote the first infinite cardinal and by ω^+ the first cardinal larger than ω.

The mappings are not necessary to be continuous. The domain of a mapping f is denoted by D_f and the range by R_f, that is if f is a mapping of a set X into a set Y, then $D_f = X$ and $R_f = Y$.

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Borel sets. We recall the notion of a Borel set (see, for example, [10]) Let X be a space. For every $\alpha \in \omega^+$ we define by induction two sets of subsets of X denoted by G_α and F_α as follows. G_0 is the set of all open subsets of X. If α is an odd (respectively, an even) ordinal, then G_α is the set of all countable intersections (respectively, of all countable unions) of elements of the set $\bigcup\{G_\beta\colon \beta \in \alpha\}$.

Similarly, F_0 is the set of all closed subsets of X. If α is an odd (respectively, an even) ordinal, then F_α is the set of all countable unions (respectively, of all countable intersections) of elements of the set $\bigcup\{F_\beta\colon \beta \in \alpha\}$.

It is easy to verify, that

$$\bigcup\{G_\alpha\colon \alpha \in \omega^+\} = \bigcup\{F_\alpha\colon \alpha \in \omega^+\}.$$

The elements of the set $\bigcup\{G_\alpha\colon \alpha \in \omega^+\}$ are called *Borel subsets* of X. A Borel subset A of X is said to be *of the additive class* (respectively, of the *multiplicative class*) $\alpha \in \omega$ if A is an element of the set G_α in the case, where α is an even (respectively, an odd) ordinal, and A is an element of the set F_α in the case, where α is an odd (respectively, an even) ordinal.

Borel mappings. A mapping f of a space X into a space Y is said to be *Borel* (see, for example, *Borel* (*measurable*) in [9] or B-*measurable* in [10]) if the set $f^{-1}(U)$ is a Borel subset of X for every open subset U of Y. If moreover $f^{-1}(U)$ is of the additive class $\alpha \in \omega^+$, then f is called a (*Borel*) *mapping of the class* α. Note that any Borel mapping is a Borel mapping of the class α for some $\alpha \in \omega$. (About some classical works on Borel mappings of the class $\alpha \in \omega^+$ see, for example, [10] and its references.)

Universal spaces. A mapping of a metric space into another metric space is said to be *isometric* or an *isometry* if it preserves the metric. Note that any isometry is a topological embedding which may be into.

A space T is said to be *topologically universal* in a class $\mathbb{P}$ of spaces if: (a) T is an element of $\mathbb{P}$ and (b) for every element X of $\mathbb{P}$ there exists a topological embedding e^X of X into T. If, moreover, the space T and the elements of $\mathbb{P}$ are metric spaces and e^X is considered to be an isometry, then T is said to be *isometrically universal* in $\mathbb{P}$.

Universal mappings. Let f and F be two mappings. A pair (i, j), where i is a topological embedding of D_f into D_F and j is a topological embedding of R_f into R_F such that $j \circ f = F \circ i$, is said to be a *topological embedding* of f into F. If, moreover, the spaces D_f, R_f, D_F, and R_F are metric and the embeddings i and j are considered to be isometries, then the pair (i, j) is called an *isometric embedding* of f into F. (Note that in the last case the mappings f and F are not necessarily to be isometries.)

A mapping F is said to be *topologically universal* in a class $\mathbb{F}$ of mappings if: (a) F is an element of $\mathbb{F}$ and (b) for every element f of $\mathbb{F}$ there exists a topological embedding (i_f, j_f) of f into F. If, moreover, the domain and range of F, as well as, the domains and ranges of all elements of $\mathbb{F}$ are metric spaces and the embedding (i_f, j_f) is considered to be an isometric embedding, then F is called *isometrically universal* in $\mathbb{F}$. (We note that in the last case the elements of $\mathbb{F}$ in general are not isometries.)

***G*-spaces.** Let G be a (multiplicative) group. By an *action of G on a set X* we mean a mapping F of $G \times X$ into X such that: (a) $F(gh, x) = F(g, F(h, x))$ for every $g, h \in G$ and $x \in X$ and (b) $F(e, x) = x$ for every $x \in X$, where e is the unit of G. In the case, where G is a (separable metrizable) topological group and X is a topological space the pair (X, F) is said to be a *G-space*.

Let (X, F) be a G-space. The pair (X, F) is said to be a *topological G-space* (respectively, a *Borel G-space of the class* $\alpha \in \omega^+$) if F is continuous (respectively, if F is Borel of the class α). (Recall that a space is called *Polish* if it is homeomorphic to a complete separable metric space. On different aspects of G-spaces (X, F^X), where G is a Polish (in particular, locally compact) topological group, X is a Polish space, and F^X is a Borel mapping, see [2] and its references.)

Universal *G*-spaces. Let G be a topological group and $\mathbb{B}$ a class of G-spaces. A G-space (T, F^T) is said to be *topologically universal* in $\mathbb{B}$ if: (a) $(T, F^T) \in \mathbb{B}$ and (b) for every element (X, F^X) of $\mathbb{B}$ there exists a topological embedding e^X of X into T such that

$$e^X \circ F^X = F^T \circ \left(id_G \times e^X\right),$$