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ScienceDirect
Topology and its Applications 155 (2008) 2128–2136
Topology and its Applications
www.elsevier.com/locate/topol

Precalibers, monolithic spaces, first countability, and homogeneity in the class of compact spaces

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Received 11 November 2006; accepted 3 April 2007

Abstract

Some new results on relationships between cardinal invariants in compacta are obtained. We establish that every non-separable compactum admits a continuous mapping onto a compactum of the weight ω_1 that has a dense non-separable monolithic subspace (Lemma 1). Lemma 1 easily implies Shapirovskij's theorem that every compactum of countable tightness and of precaliber ω_1 is separable. The lemma also opens the road to some generalizations of this statement and to other results. We also obtain new results on the structure of monolithic compacta and of homogeneous compacta. In particular, a new class of shell-homogeneous compacta is introduced and studied. One of the main results here is Theorem 31 which provides a generous sufficient condition for a homogeneous monolithic compactum to be first countable. Many intriguing open questions are formulated.

MSC: primary 54A25, 54D30; secondary 54D65

Keywords: Souslin number; Precaliber; Monolithic space; First countable; π-character; π-base; Homogeneous space; Tightness; Weight; Density; Free sequence; Compactum

Our terminology is as in [5]. "A space" in this article stands for "a Tychonoff topological space". Our notation is very close to that in [5] and [7]. However, the tightness of X is denoted by $t(X)$. Mostly, we are interested in the interplay between cardinal invariants in compact spaces. Much is already known in this direction, but even more is waiting to be discovered, we are convinced. Of particular interest seems to be Lemma 1 in Section 1. It expresses in precise terms a curious relationship between separability of compacta and monolithicity properties of their continuous images. Lemma 1 plays a key role in Section 1; in particular, a well-known theorem of Shapirovskij is easily derived from it and generalized with the help of this lemma.

A central role in Section 1 belongs to the notion of precaliber. Recall that ω_1 is said to be a *precaliber* of a space X if every uncountable family of non-empty open sets in X contains an uncountable centered subfamily. The following implications are well known and obvious: if a space X is separable, then ω_1 is a precaliber of X, and if ω_1 is a precaliber of X, then the Souslin number of X is countable. The importance of having ω_1 as a precaliber lies in the fact that this property is absolutely productive, that is, productive with respect to arbitrary families of spaces. Needless to say, neither the separability, nor the Souslin number are that nice. However, it is consistent with ZFC

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0166-8641/$ – see front matter
doi:10.1016/j.topol.2007.04.025

that ω_1 is a precaliber of every space with the countable Souslin number (see [7]). Thus, every result on spaces with precaliber ω_1 implies consistency with ZFC of the corresponding statement about spaces with the countable Souslin number.

Of course, separability is a much stronger property than precaliber ω_1. A fundamental question is, under which additional restrictions they become equivalent? B.E. Shapirovskij established in [15] that if ω_1 is a precaliber of a compactum X, and the tightness of X is countable, then X is separable. In Section 1 we generalize this important result in two ways. One of these generalizations involves the new notion of projective π-character, but the basic generalization of Shapirovskij's Theorem is Lemma 1 itself, since no restriction on the tightness or precalibers of the compactum are imposed in it whatsoever.

We also establish new facts about monolithic spaces. This notion plays an important role in both sections. A space X is said to be *monolithic* if, for every infinite subset A of X, the networkweight of the closure of A does not exceed the cardinality of A. For example, all Eberlein compacta and all Corson compacta are monolithic. A fundamental question is: when a non-empty monolithic compactum has a point of first countability? We obtain some results in this direction in Section 2. We also study in Section 2 the structure of homogeneous compacta.

One of our main results in Section 2 is the following theorem: if (CH) holds, and X is a shell-homogeneous compactum that cannot be continuously mapped onto I^{ω_1}, then X is first countable, and the cardinality of X is not greater than 2^ω (Theorem 31). The notion of a shell-homogeneous space is new, and the above result shows that it may become a useful tool on the study of homogeneity. In both sections we formulate challenging open problems.

1. Separability, precalibers, monolithicity, projective π-character in compacta

A space X will be called a *bambou* space if the weight of X is exactly ω_1, and there is a strictly increasing transfinite sequence $\eta = \{Y_\alpha\colon \alpha \in \omega_1\}$ of metrizable compact subspaces Y_α of X such that $X = \bigcup \eta = \bigcup\{Y_\alpha\colon \alpha \in \omega_1\}$.

Lemma 1. *For every non-separable compact space X, there is a continuous mapping of X onto a space Z of the weight ω_1 that has a dense bambou subspace.*

Proof. We will build up a mapping h of X to the Tychonoff cube $I^{\omega_1} = \Pi\{I_\alpha\colon \alpha < \omega_1\}$, where I_α is the usual unit interval of the real line R. To do that, we first define, by a transfinite recursion, continuous real-valued functions $f_\alpha : X \to I_\alpha$ and continuous mappings h_α of X to I^{ω_1}. We are also going to define a compact subspace K_α of X, for each $\alpha < \omega_1$.

To define the sets K_α, we will need the following concept generalizing the concept of an irreducible mapping. A mapping f of a space T onto a space M will be said to be *irreducible with respect to* a certain subset A of T if, for every closed subset B of T such that $A \subset B$ and $B \neq T$, we have $f(B) \neq M$.

The next Fact 1 is proved by an obvious standard argument using Zorn's Lemma (see the proof of the corresponding statement about irreducible mappings in [8]). So we omit the proof.

Fact 1. For every continuous mapping f of a compact space T onto a space M and every subset A of T, there is a closed subspace K of T such that $A \subset K$, $f(K) = M$, and the restriction of f to K is irreducible with respect to A.

We will also need the following:

Fact 2. If f is a continuous mapping of a compact space T onto a separable space M, and f is irreducible with respect to some separable subspace A of T, then T is also separable.

Indeed, fix a countable dense subset B of A and a countable dense subset L of M. Fix also a countable subset C of T such that $f(C) = L$ and the restriction of f to C is one-to-one. Let us show that the set $S = B \cup C$ is dense in T.

Take any non-empty open subset U of T. If U intersects A, then U intersects B, and therefore U intersects S.

It remains to consider the case when $U \cap A = \emptyset$. Then $F = T \setminus U$ is a proper closed subspace of T. Since f is irreducible with respect to A, we have $V = M \setminus f(F) \neq \emptyset$. Observe that V is open in M, since M is Hausdorff and $f(F)$ is compact. Therefore, V intersects L which implies that U intersects C. Hence, U intersects S as well, that is, S is dense in T, and T is separable. Fact 2 is established.