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Straightening Theorem for bounded Abelian groups

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ABSTRACT

We prove that every continuous function f with $f(0)=0$ between two bounded Abelian groups G and H equipped with the Bohr topology coincides with a homomorphism when restricted to an infinite subset of the domain. This extends the main results of [K. Kunen, Bohr topology and partition theorems for vector spaces, Topology Appl. 90 (1998) 97–107, D. Dikranjan, S. Watson, A solution to van Douwen's problem on the Bohr topologies, J. Pure Appl. Algebra 163 (2001) 147–158]. Moreover, we give several applications and we answer a question of [B. Givens, K. Kunen, Chromatic numbers and Bohr topologies, Topology Appl. 131 (2) (2003) 189–202].

1. Introduction

1.1. The Bohr topology and van Douwen's homeomorphism problem

Let G be an Abelian group. Following E. van Douwen [14], we will denote by $G^\#$ the group G equipped with the Bohr topology, i.e. the initial topology of G with respect to the family of all homomorphisms of G into the circle group $\mathbb{T}$.

Let us mention for future reference two fundamental properties of the Bohr topology for arbitrary Abelian groups G, H:

(a) the Bohr topology of $G \times H$ coincides with the product topology of $G^\# \times H^\#$;
(b) the Bohr topology is *functorial* (i.e., any homomorphism $H \to G$ is continuous w.r.t. the Bohr topologies of G and H); in particular, if H is a subgroup of G, then H is closed in $G^\#$ and its topology as a topological subgroup of $G^\#$ coincides with that of $H^\#$.

Call Abelian groups G and H *Bohr-homeomorphic* if $G^\#$ and $H^\#$ are homeomorphic as topological spaces [5,6]. Clearly, Bohr-homeomorphic groups have the same size, and isomorphic Abelian groups are always Bohr-homeomorphic. The following natural question was proposed by van Douwen (Question 80, [16]):

Question 1.1. Are Abelian groups of the same size always Bohr-homeomorphic?

The problem was answered negatively by Kunen [13] and independently by Dikranjan–Watson [7]. The counterexamples will be discussed below.

Towards the positive direction of van Douwen's problem, Kunen and Hart [12, Lemma 3.3.3] established the following

Fact 1.2. (See [12].) Every infinite Abelian group is Bohr-homeomorphic to its subgroups of finite index.

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They introduced the notion of *almost isomorphism* (two Abelian groups G and H are *almost isomorphic* if they have isomorphic finite index subgroups). Obviously, Fact 1.2 implies that

Corollary 1.3. *Almost isomorphic Abelian groups are always Bohr-homeomorphic.*

The following example was found by Comfort, Hernández and Trigos-Arrieta [1] to show that Bohr-homeomorphic groups need not be almost isomorphic.

Example 1.4. (See [1].) $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z} \times \mathbb{Z}$ are Bohr-homeomorphic.

1.2. Embeddings in the Bohr topology

In every pair of groups, known to provide a negative solution to van Douwen's homeomorphism problem, one of the groups is not even embeddable into the other under the Bohr topology. This motivates the study of the more general question of *embeddings* in the Bohr topology.

For every positive integer m and a cardinal κ let $\mathbb{V}_m^\kappa = \bigoplus_\kappa \mathbb{Z}_m$, where $\mathbb{Z}_m$ denotes the cyclic group of order m. It is clear that the homomorphisms $\mathbb{V}_m^\kappa \to \mathbb{Z}_m$ suffice to describe the Bohr topology of $\mathbb{V}_m^\kappa$ and a typical neighborhood of 0 in $(\mathbb{V}_m^\kappa)^\#$ is a finite-index subgroup of $\mathbb{V}_m^\kappa$ (so in this case the Bohr topology coincides with the profinite topology). See [2,4,13,7] for more details on $(\mathbb{V}_m^\kappa)^\#$.

An important step in the embedding problem for the Bohr topology was achieved by Givens and Kunen [11]. Making use of chromatic numbers of hypergraphs, they proved the following theorem characterizing those Abelian groups that admit a topological embedding into the group $(\mathbb{V}_p^\kappa)^\#$, for an infinite cardinal κ and a prime number p:

Theorem 1.5. *(See [11, Corollary 1.4].) Fix a cardinal $\kappa \geqslant \omega$, let G be an Abelian group of order κ and let p be a prime number. Then the following are equivalent*:

1. *$G^\#$ is homeomorphic to a subset of $(\mathbb{V}_p^\kappa)^\#$*;
2. *G and $\mathbb{V}_p^\kappa$ are Bohr-homeomorphic*;
3. *G and $\mathbb{V}_p^\kappa$ are almost isomorphic.*

In the same paper it is also proved that *if there exists a topological space embedding $G^\# \hookrightarrow H^\#$ and H is a bounded Abelian group, then also G must be bounded* [11, Theorem 5.1].

The above results compared to the original van Douwen's homeomorphism problem justify the following notion [5,6]:

Definition 1.6. Two Abelian groups G, H are said to be *weakly Bohr-homeomorphic* if there exist topological space embeddings $G^\# \hookrightarrow H^\#$ and $H^\# \hookrightarrow G^\#$.

In order to provide instances when two groups are weakly Bohr-homeomorphic we need also

Definition 1.7. We say that two Abelian groups G and H are *weakly isomorphic* if each one of these groups has a finite-index subgroup that is isomorphic to a subgroup of the other.

The next lemma trivially follows from Fact 1.2 and property (b) of the Bohr topologies.

Lemma 1.8. *Weakly isomorphic Abelian groups are weakly Bohr-homeomorphic.*

According to Example 1.4 the converse implication fails ($\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z} \times \mathbb{Z}$ are Bohr-homeomorphic, and yet these groups are not weakly isomorphic).

It follows from Prüfer's theorem (see [8]) that every infinite bounded group has the form $\bigoplus_{i=1}^n \mathbb{V}_{m_i}^{\kappa_i}$ for certain integers $m_i > 0$ and cardinals κ_i. For this reason, the study of the Bohr topology of the bounded Abelian groups can be focused on the groups $\mathbb{V}_m^\kappa$.

For a bounded group G, we denote by $\exp(G)$ the exponent of G (i.e., the smallest positive integer k with $kG = 0$). The *essential order* $\mathrm{eo}(G)$ of G is the smallest positive integer m with mG finite [11]. Then, $G = F \times H$, with $\mathrm{eo}(H) = \exp(H) = m$ and F finite.

The following result appeared originally in [11, Theorem 5.3] (in Section 5 we deduce this theorem from a more general result). It gave a complete solution of the embedding problem in the case of countable bounded Abelian groups:

Theorem 1.9. *(See [11].) For countably infinite bounded Abelian groups G and H, there exists an embedding $G^\# \to H^\#$ if and only if* $\mathrm{eo}(G) \mid \mathrm{eo}(H)$.