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There are no hereditary productive γ-spaces

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ABSTRACT

We show that if X is an uncountable productive γ-set [F. Jordan, Productive local properties of function spaces, Topology Appl. 154 (2007) 870–883], then there is a countable $Y \subseteq X$ such that $X \setminus Y$ is not Hurewicz.
Along the way we answer a question of A. Miller by showing that an increasing countable union of γ-spaces is again a γ-space. We will also show that λ-spaces with the Hurewicz property are precisely those spaces for which every co-countable set is Hurewicz.

1. Introduction

Our main subject in this paper is the class of (productive) γ-spaces. The term productive γ-space is misleading because it does not mean a space whose product with every γ-space is a γ-space. While the product of a productive γ-space and a γ-space is again a γ-space, it is not known if this property characterizes productively γ-spaces.

In [6], under the Continuum Hypothesis, an uncountable productive γ-subspace of $\mathbb{R}$ was constructed. The construction was based on the construction of Galvin and Miller [2], under Martin's Axiom, of a γ-space of size continuum in $\mathbb{R}$. Both examples have the property that there is a countable subset whose removal will make the space not have the Hurewicz property [2]. There do exist uncountable γ-spaces that remain γ-spaces, and hence have the Hurewicz property, when any subset is removed, see [2]. Our main purpose is to show that this is not the case for *productive* γ-space. We will prove:

Theorem 1. *If X is an uncountable productive γ-space, then there is a countable $Y \subseteq X$ such that $X \setminus Y$ is not Hurewicz.*

Along the way we will prove a general result about Fréchet filters to gain information about countable unions of γ-spaces and productive γ-spaces. In particular, we answer a question of Miller [11], which is again asked in [15]. We will also use methods from [16] to establish a result about λ-spaces with the Hurewicz property which is of independent interest.

2. Terminology

We use standard set theoretic notation. Ordinals are identified with their set of predecessors. By ω we denote the first infinite ordinal. For a set X we denote the finite and countably infinite subsets of X by $[X]^{<\omega}$ and $[X]^{\omega}$, respectively. Given sets X and Y we denote the set of all functions with domain X and range contained in Y by Y^X.

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By a *space* we mean a hereditarily Lindelöf topological space in which every open set may be written as a countable union of clopen sets. In particular, the closed sets of X are exactly the zero-sets of X. We will say *topological space* in cases where there are no assumptions about the topology.

3. Hurewicz λ-sets

Let X be a space. We say that X is *Hurewicz* [7] provided that for every sequence $(\mathcal{O}_n)_{n\in\omega}$ of open covers of X, there exist finite collections $\mathcal{V}_n \subseteq \mathcal{O}_n$ such that $X = \bigcup_{n\in\omega}\bigcap_{k\geqslant n}\bigcup \mathcal{V}_k$. Since X is Lindelöf, we may, without loss of generality, assume that the covers $\mathcal{O}_n$ in the definition of Hurewicz are countable.

Let X be a space. We say that X is a *λ-space* [9] provided that every countable subset of X is a G_δ-set.

Given $f, g \in \omega^\omega$ we write $f <^* g$ provided that $\{n \in \omega\colon\ f(n) \geqslant g(n)\}$ is finite. We say that $F \subseteq \omega^\omega$ is *bounded* provided that there is a $g \in \omega^\omega$ such that $f <^* g$ for every $f \in F$.

The following proposition was essentially proved by Hurewicz [7], see [14]:

Proposition 2. *A space X has the Hurewicz property if and only if $f[X]$ is bounded for every continuous function $f : X \to \omega^\omega$.*

Let $\overline{\omega} = \omega \cup \{\infty\}$ be the one point-compactification of ω. We say that $f \in \overline{\omega}^\omega$ is *eventually finite* provided that $\{n \in \omega\colon$ $f(n) = \infty\}$ is finite. Let $\mathbb{EF} \subseteq \overline{\omega}^\omega$ be the set of all eventually finite functions. We say that $f : X \to \mathbb{EF}$ is *almost-finite* provided that $\{x \in X\colon\ \infty \in f(x)[\omega]\}$ is countable. We say that $F \subseteq \mathbb{EF}$ is *bounded* provided that there is a $g \in \omega^\omega$ such that $\{n \in \omega\colon\ f(n) \geqslant g(n)\}$ is finite for every $f \in F$. We say that $S \subseteq X$ is *co-countable* provided that $X \setminus S$ is countable.

The following theorem gives a characterization of Hurewicz λ-spaces. Its proof follows the method developed in [16].

Theorem 3. *Let X be a space. The following conditions are equivalent*:

(a) *$F[X]$ is bounded for every continuous almost-finite function $F : X \to \mathbb{EF}$,*
(b) *X is Hurewicz and X is a λ-set, and*
(c) *every co-countable subset of X is Hurewicz.*

The proof of Theorem 3 consists of the following three lemmas.

Lemma 4. *If every co-countable subset of a space X is Hurewicz, then $F[X]$ is bounded in $\mathbb{EF}$ for every continuous almost-finite function $F : X \to \mathbb{EF}$.*

Proof. Let $F : X \to \mathbb{EF}$ be continuous and almost-finite. For each $n \in \omega$ let $G_n = \{g \in F[X]\colon\ g(k) \neq \infty$ for all $k \geqslant n\}$. Notice that $X_n = F^{-1}(G_n)$ is a co-countable set for every $n \in \omega$. So, X_n is Hurewicz for every $n \in \omega$.

For each $n \in \omega$. Let $T_n : G_n \to \mathbb{EF}$ be the shift transformation $T_n(g)(k) = g(n + k)$. Since T_n is continuous for every $n \in \omega$, we have, by Proposition 2, that $T_n[F[X_n]]$ is bounded in ω^ω for each $n \in \omega$. So, $F[X_n]$ is bounded in $\mathbb{EF}$ for each $n \in \omega$. Since $X = \bigcup_{n\in\omega} X_n$, $F[X]$ is bounded in $\mathbb{EF}$. □

Lemma 5. *Let X be a space. If $F[X]$ is bounded in $\mathbb{EF}$ for every continuous almost-finite function $F : X \to \mathbb{EF}$, then X is a Hurewicz λ-space.*

Proof. Let $f : X \to \omega^\omega$ be continuous. Since f is almost-finite, $f[X]$ is bounded in ω^ω. By Proposition 2, X is Hurewicz.

We now show that X is a λ-space. Let G be a co-countable subset of X. Let $\{x_n\colon\ n \in \omega\}$ be an enumeration of $X \setminus G$.

Since X is a space, closed sets are zero sets. So, we may define for each $n \in \omega$ a continuous function $f_n : X \to \overline{\omega}$ so that $f_n^{-1}(\infty) = \{x_n\}$. Define $F : X \to \mathbb{EF}$ by $F(x)(n) = f_n(x)$. Notice that F is continuous and almost-finite. Thus, $F[X]$ is bounded in $\mathbb{EF}$ by some $h \in \omega^\omega$. Notice that

$$G = \{x \in X\colon\ \infty \notin F(x)[\omega]\} = F^{-1}\big(\{g \in \omega^\omega\colon\ g <^* h\}\big).$$

Notice that $\{g \in \omega^\omega\colon\ g <^* h\}$ is an F_σ-subset of $\mathbb{EF}$. It follows that G is an F_σ-subset of X. Thus, X is a λ-space. □

Lemma 6. *If a space X is a Hurewicz λ-space, then $F[X]$ is bounded in $\mathbb{EF}$ for every continuous almost-finite function $F : X \to \mathbb{EF}$.*

Proof. Let $F : X \to \mathbb{EF}$ be a continuous almost-finite function. Let $G = X \setminus E$ where $E = \{x \in X\colon\ \infty \in F(x)[\omega]\}$. Since E is countable, G is an F_σ. Since closed subsets of Hurewicz spaces are closed and the Hurewicz property is preserved by countable unions, G is Hurewicz. Thus, $F[G]$ is bounded in ω^ω. Since $F[E]$ is countable, it follows that $F[X]$ is bounded in $\mathbb{EF}$. □