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# Irregularity ☆

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**Abstract**

Regular and irregular pretopologies are studied. In particular, for every ordinal there exists a topology such that the series of its partial (pretopological) regularizations has length of that ordinal. Regularity and topologicity of special pretopologies on some trees can be characterized in terms of sets of intervals of natural numbers, which reduces studied problems to combinatorics.


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## 1. Introduction

By a *convergence* we understand a relation $x \in \lim \mathcal{F}$, between filters $\mathcal{F}$ and points $x$, such that $\mathcal{F} \subset \mathcal{G}$ implies $\lim \mathcal{F} \subset \lim \mathcal{G}$, and for which the principal ultrafilter of $x$ converges to $x$ for every point $x$. A convergence $\zeta$ is *finer* than a convergence $\xi$ (in symbols, $\zeta \geqslant \xi$) if $\lim_\zeta \mathcal{F} \subset \lim_\xi \mathcal{F}$ for each filter $\mathcal{F}$. A map $f$ from a convergence space to another is *continuous* provided that $f(\lim \mathcal{F}) \subset \lim f(\mathcal{F})$ for every filter $\mathcal{F}$.[1] The class of convergences is a category (with continuous maps as morphisms). A convergence is *Hausdorff* if the limit of every filter is at most a singleton.

The notion of regularity was generalized from topological to convergence spaces in two ways, by Fischer [13] and by Grimeisen [15,16]. A convergence is *regular* (in the sense of Fischer) if the limit of a filter $\mathcal{F}$ is included in the limit of the filter generated by the family of the adherences of the elements of $\mathcal{F}$. The definitions of Fischer and Grimeisen coincide for pseudotopological spaces, and *a fortiori* for pretopological spaces, which are the framework of this paper.[2]

Regular convergences form a concretely reflective subcategory of the category of convergences; we denote its reflector by $R$. In particular, for every convergence $\xi$ there exists a regular convergence $R\xi$, which is the finest among the regular convergences that are coarser than $\xi$. The convergence $R\xi$ is the *regular reflection* of $\xi$ (the *regularization* of $\xi$).

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1 We denote by $f(\mathcal{F})$ the filter generated by $\{f(F)\colon F \in \mathcal{F}\}$.
2 Pseudotopologies and pretopologies are subclasses of convergences; we will define them in Preliminaries.

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To our knowledge, there exists no explicit description of the filters that converge in $R\xi$ in terms of those convergent in $\xi$. It is possible however to define, explicitly and simply, a *partial regularization* $r\xi$ of $\xi$ so that $\xi \geqslant r\xi \geqslant R\xi$ for every convergence $\xi$, and a convergence $\tau$ is regular if and only if $\xi \leqslant r\xi$.[3] Moreover $r$ is a concrete functor, and for each convergence $\xi$ there is a least ordinal $\alpha$ (the *irregularity* of $\xi$) such that $R\xi$ is equal to the $\alpha$th iteration[4] of $r$ applied to $\xi$.

In this paper we show that for each ordinal $\alpha$ there exists a Hausdorff pretopology the irregularity of which is precisely $\alpha$. Our result is more precise (and our construction is much simpler) than that of Kent and Richardson [18,19] who proved that for every ordinal $\beta$ there exists a pretopology $\xi$ such that $\beta$ is the least ordinal for which $(r^{\omega})^{\beta}\xi = R\xi$.

We call an element $x$ *regular* for a convergence $\xi$ if $x \in \lim_{r\xi} \mathcal{F}$ implies $x \in \lim_{\xi} \mathcal{F}$ for every filter $\mathcal{F}$, and *irregular* otherwise. We witness an interesting phenomenon of "propagation of irregularities" concerning the regularity of elements: an element can be regular for a convergence $\xi$ but irregular for its partial regularization $r\xi$, which, by construction, is "more regular" than $\xi$. This observation leads to a notion of irregularity spectrum.

The *irregularity* of $x$ with respect to $\xi$ is the least ordinal $\beta$ such that $x$ is regular for $r^{\beta}\xi$. The *irregularity spectrum* of an element $x$ with respect to a convergence $\xi$ is the set of ordinals $\alpha$ for which $x$ is irregular for $r^{\alpha}\xi$. Consequently, an element is irregular if and only if 0 is in its spectrum. It is amazing that for every subset $A$ of an ordinal, one can construct a Hausdorff pretopology such that the irregularity spectrum of an element with respect to this convergence is precisely $A$.

Study of regularity (and irregularity) of some special pretopologies on sequential trees (*standard pretopologies*) led us to a concept of *states* (sets of intervals of an ordinal). Each standard pretopology is completely determined by its state, and the functors $r$, $R$ are transferred to the space of states. In this way, each investigation concerning regularity of such a pretopology can be reduced to a combinatorial problem concerning states.

We have observed that an element $x$ of a pretopology of *countable character*[5] is irregular (thus of irregularity $\geqslant 1$), then there exists a homeomorphic embedding "at $x$" of an irregular standard pretopology (on a tree of rank 2). On the other hand, the fact that an element $x$ is of irregularity 2 does not imply the existence of a homeomorphic embedding "at $x$" of an irregular standard pretopology on a tree of rank 3.

This discovery led us to a concept of *ramified standard pretopologies* and to our main result that *if $x$ is an element of finite irregularity of a pretopology of countable character, then there is a homeomorphic embedding "at $x$" of a ramified standard pretopology of the same irregularity.*

## 2. Preliminaries

Families $\mathcal{F}, \mathcal{H}$ (of subsets of a given set) *mesh* (in symbols, $\mathcal{F} \# \mathcal{H}$) if $F \cap H \neq \emptyset$ for every $F \in \mathcal{F}$ and each $H \in \mathcal{H}$. A systematic use of the operation # in conjunction with other operations, like that of contour, has led to a versatile calculus (see, for example, [8,9,3,11,4]). The operation # is related to the notion of *grill* $\mathcal{H}^{\#}$ of a family $\mathcal{H}$, which was defined by Choquet [1] as $\mathcal{H}^{\#} = \bigcap_{H \in \mathcal{H}} \{G\colon\ G \cap H \neq \emptyset\}$ (denoted also $\sec(\mathcal{H})$ in [17]); of course,

$$\mathcal{F} \# \mathcal{H} \iff \mathcal{F} \subset \mathcal{H}^{\#} \iff \mathcal{H} \subset \mathcal{F}^{\#}.$$

The *adherence* of a filter $\mathcal{H}$ with respect to a convergence $\xi$ is defined by

$$\operatorname{adh}_{\xi} \mathcal{H} = \bigcup_{\mathcal{F} \# \mathcal{H}} \lim_{\xi} \mathcal{F}.$$

In particular, $\operatorname{adh}_{\xi} H$ denotes the adherence of the principal filter of $H$. If $\mathcal{F}$ is a filter on the underlying set $|\xi|$ of a convergence $\xi$, then the symbol $\operatorname{adh}_{\xi}^{\natural} \mathcal{F}$ denotes the filter generated by $\{\operatorname{adh}_{\xi} F\colon\ F \in \mathcal{F}\}$. The infimum $\mathcal{V}_{\xi}(x)$ of all filters that converge to $x$, is called the *vicinity filter* of $x$ with respect to $\xi$.

A convergence $\xi$ is *regular* (in the sense of Fischer) if

$$\lim_{\xi} \mathcal{F} \subset \lim_{\xi} \left(\operatorname{adh}_{\xi}^{\natural} \mathcal{F}\right) \tag{2.1}$$

[3] Kent and Richardson [18,19] introduced another functor of partial regularization, which in our terminology is equal to $r^{\omega}$.

[4] To be defined later.

[5] Also called *first-countable*.