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On the signature of a Lefschetz fibration coming from an involution ☆

Ki-Heon Yun

Department of Mathematics, Konkuk University, 1 Hwayang-dong, Gwangjin-gu, Seoul 143-701, Republic of Korea

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Abstract

In this article we show that the signature of a Lefschetz fibration coming from a special involution as a product of right-handed Dehn twists depends only on the number of genus on the involution axis. We investigate the geography of such Lefschetz fibrations and we identify it with a blow up of a ruled surface. We also get a geography of the Lefschetz fibration coming from a finite order element of mapping class group as a composition of two special involutions.

MSC: 57N13; 57R17

Keywords: Lefschetz fibration; Signature; Mapping class group

1. Introduction

The study of symplectic topology in dimension four is closely related to the study of Lefschetz fibration over S^2 which is determined by monodromy factorization. A relation in the mapping class group as a product of right-handed Dehn twists gives a monodromy factorization of a Lefschetz fibration.

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E-mail address: kyun@member.ams.org (K.-H. Yun).

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Korkmaz [7] found a relation in the mapping class group involving $2g+4$ (resp., $2g+10$) right-handed Dehn twists when the genus g of the surface is even (resp., odd). Gurtas [5,6] generalized it and he found an involution ϕ of a Riemann surface Σ_g as a product of $g+3h+2$ right-handed Dehn twists where h is the number of genus on the involution axis. Gurtas asked whether the signature $\sigma(X_{\phi^2})$ of the Lefschetz fibration $X_{\phi^2}\to S^2$ is $-4(h+1)$. It is known that the signature is -4 for $h=0$ case [7] but it is not known for $h\geqslant 1$.

The article is organized as follows. In Section 2, we briefly review a Lefschetz fibration and a right-handed Dehn twists expression of an involution of type $(l,2k,r)$ (Definition 4) which was introduced by Gurtas. We show that for a fixed $g=l+2k+r$ and $h=l+r$, all involutions of type $(l,2k,r)$ are related by a sequence of Hurwitz moves and conjugations.

In Section 3, we get a signature formula for the Lefschetz fibration coming from an involution of type $(l,2k,r)$. Furthermore we also identify such a Lefschetz fibration.

Theorem 1. *Let $\phi:\Sigma_g\to\Sigma_g$ be the right-handed Dehn twists expression of an involution of type $(l,2k,r)$. Let $X_\phi\to D^2$ be the Lefschetz fibration corresponding to ϕ and $X_{\phi^2}\to S^2$ be the Lefschetz fibration corresponding to ϕ^2. Then $\sigma(X_{\phi^2})=-4(h+1)$ and $\sigma(X_\phi)=-2(h+1)$. Furthermore, X_{ϕ^2} is diffeomorphic to $(\Sigma_k\times S^2)\#4(l+r+1)\overline{\mathbb{CP}}^2$.*

In Section 4, we study a signature formula of a Lefschetz fibration over S^2 coming from a finite order element of a mapping class group as a composition of two involutions. As an application, we study a p-fold cyclic covering $\pi_p:\Sigma_{p(g-1)+1}\to\Sigma_g$ of the Riemann surface Σ_g and a right-handed Dehn twists expression ψ_p of the $\frac{2\pi}{p}$ rotation map on $\Sigma_{p(g-1)+1}$ which sends the ith handle to the $(i+1)$st handle of $\Sigma_{p(g-1)+1}$. When p is odd, ψ_p is a composition of two involutions of type $(g,(p-1)(g-1),0)$. If p is even, then ψ_p is a composition of an involution of type $(g,(p-2)(g-1),(g-1))$ and an involution of type $(1,p(g-1),0)$.

Theorem 2. *The Lefschetz fibration $X_{\psi_p^p}\to S^2$ is a simply connected 4-manifold which has the following topological invariants*:

$$c_1^2(X_{\psi_p^p})=4p(p-2)(g-1)\quad\textit{and}\quad\chi_h(X_{\psi_p^p})=\frac{1}{8}\left\{c_1^2(X_{\psi_p^p})+4p(g+1)\right\}.$$

2. Preliminaries

Definition 3. Let X be a compact, oriented smooth 4-manifold. A (smooth) Lefschetz fibration is a proper smooth map $\pi:X\to B$ where B is a compact connected oriented surface and $\pi^{-1}(\partial B)=\partial X$ such that

(1) the set of critical points $C=\{p_1,p_2,\ldots,p_n\}$ of π is non-empty and lies in $\operatorname{int}(X)$ and π is injective on C;
(2) about each p_i and $\pi(p_i)$, there are local complex coordinate charts agreeing with the orientations of X and B such that π can be expressed as $\pi(z_1,z_2)=z_1^2+z_2^2$.