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On the Steinhaus property in topological groups

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Abstract

Let G be a locally compact Abelian group and μ a Haar measure on G. We prove: (a) If G is connected, then the complement of a union of finitely many translates of subgroups of G with infinite index is μ-thick and everywhere of second category. (b) Under a simple (and fairly general) assumption on G, for every cardinal number $\mathfrak{m}$ such that $\aleph_0 \leqslant \mathfrak{m} \leqslant |G|$ there is a subgroup of G of index $\mathfrak{m}$ that is μ-thick and everywhere of second category. These results extend theorems by Muthuvel and Erdős–Marcus, respectively. (b) also implies a recent theorem by Comfort–Raczkowski–Trigos stating that every nondiscrete compact Abelian group G admits $2^{|G|}$-many μ-nonmeasurable dense subgroups.

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1. Introduction

Starting point of our analysis is the following question. Let G be an Abelian group, $H_1, \ldots, H_n$ subgroups of G of infinite index, and $x_1, \ldots, x_n \in G$. How "large" is the complement $Z^c := G \setminus Z$ of $Z := \bigcup_{i=1}^{n}(x_i + H_i)$?

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Neumann's lemma [7, Lemma 7.3] states that Z^c is nonempty. This purely algebraic fact is sharpened in Theorem 2.2: we show that Z^c contains some translate of a subgroup of G with index $\aleph_0 := |\mathbb{N}|$. The problem of "estimating" Z^c gains a particular interest in a topological setting, namely when G is also supposed to be a Hausdorff topological group. In the case where $G = \mathbb{R}$ and $x_1 = \cdots = x_n = 0$, Muthuvel [16, Theorem 2] proves that Z^c is everywhere of second category. We give in Theorem 4.6 a substantial generalization of his result, in particular replacing $\mathbb{R}$ with an arbitrary connected Baire group. Theorem 4.7 represents the measure theoretical counterpart of this result: if G is locally compact and connected, then Z^c is thick with respect to a Haar measure on G. (For the definitions of everywhere of second category and thick sets, see Section 3.1.)

The "dual" results just cited are by no means accidental: in fact, the basic ingredient for proving Theorems 4.6 and 4.7 turns out to be the so-called Steinhaus property (see Definition 3.2), which we study in a more general framework in Section 3.2. As a first illustration of the advantages of our unified approach, we stress that Theorems 4.6 and 4.7 are both "extracted" from the abstract Theorem 4.1.

Let us now describe the second problem addressed in this paper. Erdős and Marcus prove in [6] that for every cardinal number $\mathfrak{m}$ such that $\aleph_0 \leqslant \mathfrak{m} \leqslant \mathfrak{c} := |\mathbb{R}|$ there exists a partition of $\mathbb{R}^n$ into $\mathfrak{m}$-many thick congruent subsets (i.e. each a translate of the other; they are even chosen to be cosets of a single subgroup). They also observe that there is an analogous partition of $\mathbb{R}^n$ into congruent, everywhere of second category subsets. In Theorem 4.8 we extend their result to a large class of locally compact Abelian groups (namely, those satisfying the condition $(*)$ introduced in Proposition 4.4). As above, we emphasize that Theorem 4.8 is easily obtained from the more general Theorem 4.3, the latter being a further consequence of the Steinhaus property (and of the algebraic Theorem 2.3).

A direct consequence of Theorem 4.8 is Corollary 4.9. It states, in particular, that every nondiscrete locally compact Abelian group G satisfying the condition $(*)$ (such as any locally compact group which is connected or σ-compact) admits $2^{|G|}$-many nonmeasurable dense subgroups of cardinality $|G|$. In the compact case, this has recently been obtained by Comfort, Raczkowski, and Trigos-Arrieta (see [3, Theorem 3.4]). While their proof strongly depends on the duality theory for locally compact Abelian groups, our proof is based on the more elementary algebraic Theorem 2.3.

Throughout the paper, G stands for an additively written group. (For our main results G shall be assumed to be Abelian.) If H is a subgroup of G, then $|G : H|$ denotes the index of H in G, i.e. $|G : H| := |G/H|$. For $X \subseteq G$, by $\langle X\rangle$ we denote the subgroup of G generated by X. The symbol $\mathcal{P}(G)$ denotes the power set of G.

2. Algebraic tools

The two algebraic theorems in this section are needed for our main results contained in Section 4. Proposition 2.1 below is applied by Hewitt and Ross to show that every infinite compact Abelian group contains nonmeasurable subgroups. We use it in the proof of Theorem 2.2.