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Proximity and distality via Furstenberg families ☆

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Abstract

In this paper proximity, distality and recurrence are studied via Furstenberg families. A new proof of some classical results on the conditions when a proximal relation is an equivalence one is given. Moreover, for a family $\mathcal{F}$, $\mathcal{F}$-almost distality and $\mathcal{F}$-semi-distality are defined and characterized. As an application a new characterization of PI-flows is obtained.

MSC: 37B05; 37B20

Keywords: Proximity; Distality; Recurrence; Furstenberg family

1. Introduction

Throughout this paper a *topological dynamical system* (TDS for short) is a pair (X, T), where X is a nonvoid compact metric space with a metric d and T is a continuous surjective map from X to itself. We use $\mathbb{Z}$ to denote the set of integers, $\mathbb{Z}_+$ the set of non-negative integers and $\mathbb{N}$ the set of natural numbers. Let $\mathrm{Trans}_T = \{x\colon \omega T(x) = X\}$, where $\omega T(x)$ is the ω-limit set of x. Say (X, T) is *transitive* if $\mathrm{Trans}_T \neq \emptyset$. In fact, Trans_T is a dense G_δ set when it is not empty. Say (X, T) is *minimal* if X is the only non-empty closed and invariant subset, and $x \in X$ is a *minimal point* if it belongs to some minimal subsystem

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of X. Sometimes we need to consider the case when the phase space is an arbitrary compact Hausdorff space, and in this case we define minimality in the same way.

Classically, one way of studying a TDS is to consider the asymptotic behavior of pairs of points. A pair $(x, y) \in X \times X = X^2$ is said to be *proximal* if $\liminf_{n\to+\infty} d(T^n x, T^n y) = 0$ and the one with $\lim_{n\to+\infty} d(T^n x, T^n y) = 0$ is said to be *asymptotic*. If in addition $x \neq y$, then the pair (x, y) is said to be *proper*. The sets of proximal pairs and asymptotic pairs of (X, T) are denoted by $P(X, T)$ and $\mathrm{Asym}(X, T)$ respectively. $P(X, T)$ is a reflexive, symmetric, T-invariant relation, but in general not transitive or closed [3–5].

A pair $(x, y) \in X^2$ which is not proximal is said to be *distal*. A pair is a *Li–Yorke pair* if it is proximal but not asymptotic. $x \in X$ is a *recurrent point* if there is an increasing sequence $\{n_i\}$ of $\mathbb{N}$ with $T^{n_i} x \to x$. A pair $(x, y) \in X^2 \setminus \Delta_X$ is a *strong Li–Yorke pair* if it is proximal and is a recurrent point of $T \times T$. It is easy to check that a strong Li–Yorke pair is a Li–Yorke pair. A system without proper proximal pairs (Li–Yorke pairs, strong Li–Yorke pairs) is called *distal* (*almost distal*, *semi-distal* respectively). It is clear that a distal system is almost distal and an almost distal system is semi-distal.

A beautiful characterization of distality was given by R. Ellis using so-called enveloping semigroup. Given a TDS (X, T) its *enveloping semigroup* $E(X, T)$ is defined as the closure of the set $\{T^n\colon n \in \mathbb{Z}_+\}$ in X^X (with its compact, usually non-metrizable, pointwise convergence topology). Ellis showed that a TDS (X, T) is distal iff $E(X, T)$ is a group iff every point in X^2 is minimal [9]. The notion of almost distal was first introduced by Blandchard etc. [7]. Let the *adherence semigroup* $\mathcal{H}(X, T)$ be $\limsup\{T^n\} = \bigcap_{k=1}^{\infty} \overline{\{T^n\colon n = k, k+1, \ldots\}} \subset X^X$. They showed that a TDS (X, T) is almost distal iff $(\mathcal{H}(X, T), T)$ is minimal iff every ω-limit set in $(X^2, T \times T)$ is minimal. Recently, Akin etc. studied distality concepts for Ellis actions [1]. They defined a system without strong Li–Yorke pairs to be *semi-distal*, i.e. every $(x, y) \in X^2$ which is both proximal and recurrent is in the diagonal. They gave an elegant characterization of semi-distality via the enveloping semigroup, namely they showed that a TDS is semi-distal iff every idempotent in $\mathcal{H}(X, T)$ is minimal iff every recurrent point in $(X^2, T \times T)$ is minimal.

In this paper we investigate the proximal relation from the viewpoint of Furstenberg families and give a new proof of some classical results on the conditions when a proximal relation is an equivalence one. By using the family notion our proofs become simpler and clearer. Moreover, family machinery is applied to describe family versions of distality, almost distality and semi-distality. Different notions are unified by this family viewpoint, and in particular, we show that a minimal PI-flow can be viewed as some kind of semi-distal one. By applying the structure theorems of some special minimal systems, we can give a negative answer to a conjecture by Blanchard etc. [7] on the structure of minimal almost distal systems.

2. Preliminary

Firstly we introduce some notations related to a family (for details see [2,10]). Let $\mathcal{P} = \mathcal{P}(\mathbb{Z}_+)$ be the collection of all subsets of $\mathbb{Z}_+$. A subset $\mathcal{F}$ of $\mathcal{P}$ is a *family*, if it is hereditary upwards, i.e. $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$. A family $\mathcal{F}$ is *proper* if it is a proper subset of $\mathcal{P}$, i.e. neither empty nor all of $\mathcal{P}$. It is easy to see that a family $\mathcal{F}$