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A new bound on the cardinality of homogeneous compacta

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Abstract

We show (in ZFC) that if X is a compact homogeneous Hausdorff space then $|X| \leqslant 2^{t(X)}$, where $t(X)$ denotes the tightness of X. It follows that under *GCH* the character and the tightness of such a space coincide.

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1. Introduction

A space X is said to have *countable tightness* (in symbols $t(X) = \aleph_0$) if whenever $A \subseteq X$ and $x \in \overline{A}$, there is a countable $B \subseteq A$ such that $x \in \overline{B}$. A space X is *homogeneous* if for every $x, y \in X$ there is a homeomorphism f of X onto X with $f(x) = y$. It is known that any compact space of countable tightness contains a point with character at most $2^{\aleph_0}$; if the space is also homogeneous then it follows that $|X| \leqslant 2^{2^{\aleph_0}}$. In [1], Arkhangel'skiĭ asked if in fact $|X| \leqslant 2^{\aleph_0}$ for any such space; he later conjectured a positive answer to this question (see [4]). A well-known result of A. Dow (see [5]) states that under *PFA* any compact space of countable tightness contains a point of countable character; from this it follows that

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Arkhangel'skiĭ's conjecture is true under *PFA*. A related result of Arkhangel'skiĭ (see [1, 2]) states that if X is a sequential (and hence of countable tightness) compact homogeneous space then $|X| \leqslant 2^{\aleph_0}$.

The main goal of this paper is to give a proof in ZFC of Arkhangel'skiĭ's conjecture. In fact we will prove (Theorem 3.2) that the same is true for higher cardinals (i.e. $|X| \leqslant 2^{t(X)}$ for any compact homogeneous X). This generalizes a result of M. Ismail (see [6]), who showed that $|X| \leqslant 2^{t(X)}$ for compact homogeneous X satisfying the countable chain condition. As a corollary of our result we also confirm a conjecture of I. Juhász, P. Nyikos and Z. Szentmiklóssy (see [7]), stating that it is consistent that every homogeneous T_5 compactum is first countable.

Our main tool will be the "Elementary Submodels technique": Given a topological space (X, τ), one lets M be an elementary submodel of $H(\theta)$ (the set of all sets of hereditary cardinality less than θ) for a "large enough" regular cardinal θ. Usually one asks for M to be "small" and to contain X and τ as elements. Then one uses closure properties of M to get results about $X \cap M$, $\tau \cap M$ and ultimately about (X, τ). A model M is said to be *κ-closed* if any κ-sequence of elements of M is in M (i.e. $M^\kappa \subseteq M$). For more details and a good introduction to the technique see [5]. Let us just say that in each specific application, one takes θ large enough for $H(\theta)$ to contain all sets of interest in the context under discussion. In this sense we will just say that $M \prec \mathbf{V}$. In Section 2 we prove some basic facts in the context of elementary submodels of compact spaces with $t(X) \leqslant \kappa$; we also give an answer (Theorem 2.2) to a question of L.R. Junqueira and F.D. Tall.

We assume all spaces to be Hausdorff. If $A \subseteq X$ we write $\overline{A}$ for the topological closure of A in X. As usual $[X]^\kappa$ is the set of all subsets of X of cardinality κ and $[X]^{\leqslant\kappa}$ is the set of all subsets of X of size no more than κ. A set $A \subseteq X$ is called a *G_κ-set* or a *G_κ-subset* of X if it is the intersection of no more than κ open subsets of X.

2. Elementary submodels

Let κ be an infinite cardinal and fix a compact space (X, τ) with $t(X) \leqslant \kappa$. Fix a κ-closed $M \prec \mathbf{V}$ with $X, \tau \in M$. Let $Z = \overline{X \cap M} \subseteq X$ with the subspace topology.

One of the main goals of this section is to show that Z is a retract of X. The following result suggests what the retraction is going to be.

Lemma 2.1. *For every $x \in X$ there is a $q_x \in Z$ such that for all $U \in \tau \cap M$ either $q_x \notin U$ or $x \in U$.*

Proof. Fix $x \in X$ and assume there is not such a q_x. Then for each $q \in Z$ we can fix a $U_q \in \tau \cap M$ such that $q \in U_q$ and $x \notin U_q$. Since Z is compact we get that $Z \subseteq \bigcup_{q\in Q} U_q$ for some finite $Q \subseteq Z$. On the other hand, $x \notin \bigcup_{q\in Q} U_q \in M$ so by elementarity there is an $x' \in (X \cap M) \setminus \bigcup_{q\in Q} U_q$ which is impossible. □

Just by elementarity and the fact that X is Hausdorff, it is immediate that $\tau \cap M$ separates points in $X \cap M$. We prove now that in fact $\tau \cap M$ separates points in Z.