Annals of Pure and Applied Logic 168 (2017) 19–36

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

A classification of orbits admitting a unique invariant measure

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ARTICLE INFO

Article history:
Received 19 July 2016
Accepted 15 August 2016
Available online 24 August 2016

MSC:
03C98
37L40
60G09
20B27

Keywords:
Invariant measure
High homogeneity
Unique ergodicity

ABSTRACT

We consider the space of countable structures with fixed underlying set in a given countable language. We show that the number of ergodic probability measures on this space that are S_∞-invariant and concentrated on a single isomorphism class must be zero, or one, or continuum. Further, such an isomorphism class admits a unique S_∞-invariant probability measure precisely when the structure is highly homogeneous; by a result of Peter J. Cameron, these are the structures that are interdefinable with one of the five reducts of the rational linear order $(\mathbb{Q}, <)$.

1. Introduction

A countable structure in a countable language can be said to admit a random symmetric construction when there is a probability measure on its isomorphism class (of structures having a fixed underlying set) that is invariant under the logic action of S_∞. Ackerman, Freer, and Patel [1] characterized those structures admitting such invariant measures. In this paper, we further explore this setting by determining the possible numbers of such ergodic invariant measures, and by characterizing when there is a unique invariant measure.

A dynamical system is said to be uniquely ergodic when it admits a unique, hence necessarily ergodic, invariant measure. In most classical ergodic-theoretic settings, the dynamical system consists of a measure space along with a single map, or at most a countable semigroup of transformations; unique ergodicity has

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http://dx.doi.org/10.1016/j.apal.2016.08.003
0168-0072/

been of longstanding interest for such systems. In contrast, unique ergodicity for systems consisting of a larger space of transformations (such as the automorphism group of a structure) has been a focus of more recent research, notably that of Glasner and Weiss [12], and of Angel, Kechris, and Lyons [2].

When studying continuous dynamical systems, one often considers minimal flows, i.e., continuous actions on compact Hausdorff spaces such that each orbit is dense; [2] examines unique ergodicity in this setting. In the present paper, we are interested in unique ergodicity of actions where the underlying space need not be compact and there is just one orbit: We characterize when the logic action of the group S_∞ on an orbit is uniquely ergodic.

Any transitive S_∞-space is isomorphic to the action of S_∞ on the isomorphism class of a countable structure, restricted to a fixed underlying set. The main result of [1] states that such an isomorphism class admits at least one S_∞-invariant measure precisely when the structure has trivial definable closure. Here we characterize those countable structures whose isomorphism classes admit *exactly one* such measure, and show via a result of Peter J. Cameron that the five reducts of $(\mathbb{Q}, <)$ are essentially the only ones. Furthermore, if the isomorphism class of a countable structure admits more than one S_∞-invariant measure, it must admit continuum-many ergodic such measures.

1.1. Motivation and main results

In this paper we consider, for a given countable language L, the collection of countable L-structures having the natural numbers $\mathbb{N}$ as underlying set. This collection can be made into a measurable space, denoted Str_L, in a standard way, as we describe in Section 2.

The group S_∞ of permutations of $\mathbb{N}$ acts naturally on Str_L by permuting the underlying set of elements. This action is known as the *logic action* of S_∞ on Str_L, and has been studied extensively in descriptive set theory. For details, see [4, §2.5] or [11, §11.3]. Observe that the S_∞-orbits of Str_L are precisely the isomorphism classes of structures in Str_L.

By an *invariant measure* on Str_L, we will always mean a Borel probability measure on Str_L that is invariant under the logic action of S_∞. We are specifically interested in those invariant measures on Str_L that assign measure 1 to a single orbit, i.e., the isomorphism class in Str_L of some countable L-structure $\mathcal{M}$. In this case we say that the orbit of $\mathcal{M}$ *admits* an invariant measure, or simply that $\mathcal{M}$ admits an invariant measure.

When a countable structure $\mathcal{M}$ admits an invariant measure, this measure can be thought of as providing a symmetric random construction of $\mathcal{M}$. The main result of [1] describes precisely when such a construction is possible: A structure $\mathcal{M} \in \mathrm{Str}_L$ admits an invariant measure if and only if definable closure in $\mathcal{M}$ is trivial, i.e., the pointwise stabilizer in $\mathrm{Aut}(\mathcal{M})$ of any finite tuple fixes no additional elements. But even when there *are* invariant measures concentrated on the orbit of $\mathcal{M}$, it is not obvious how many different ones there are.

If an orbit admits at least two invariant measures, there are trivially always continuum-many such measures, because a convex combination of any two gives an invariant measure on that orbit, and these combinations yield distinct measures. It is therefore useful to count instead the invariant measures that are not decomposable in this way, namely the *ergodic* ones. It is a standard fact that the invariant measures on Str_L form a simplex in which the ergodic invariant measures are precisely the *extreme* points, i.e., those that cannot be written as a nontrivial convex combination of invariant measures. Moreover, every invariant measure is a mixture of these extreme invariant measures. (For more details, see [16, Lemma A1.2 and Theorem A1.3] and [22, Chapters 10 and 12].) Thus when counting invariant measures on an orbit, the interesting quantity to consider is the number of ergodic invariant measures.

Many natural examples admit more than one invariant measure. For instance, consider the Erdős–Rényi [7] construction $G(\mathbb{N}, p)$ of the Rado graph, a countably infinite random graph in which edges have indepen-