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Cardinal characteristics at κ in a small $\mathfrak{u}(\kappa)$ model

A.D. Brooke-Taylor [a], V. Fischer [b,*], S.D. Friedman [b], D.C. Montoya [b]

[a] School of Mathematics, University of Bristol, University Walk, Bristol, BS8 1TW, UK
[b] Kurt Gödel Research Center, Währinger Strasse 25, 1090 Vienna, Austria

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We provide a model where $\mathfrak{u}(\kappa) < 2^\kappa$ for a supercompact cardinal κ. [10] provides a sketch of how to obtain such a model by modifying the construction in [6]. We provide here a complete proof using a different modification of [6] and further study the values of other natural generalizations of classical cardinal characteristics in our model. For this purpose we generalize some standard facts that hold in the countable case as well as some classical forcing notions and their properties.

1. Introduction

Cardinal characteristics on the Baire space ω^ω have been widely studied and understood. Since 1995 with the Cummings–Shelah paper [5], the study of the generalization of these cardinal notions to the context of uncountable cardinals and their interactions has been developing. By now, there is a wide literature on this topic. Some key references (at least for the purposes of this paper) are [2,5] and [14].

In [6] Džamonja and Shelah construct a model with a universal graph at the successor of a strong limit singular cardinal of countable cofinality. A variant of this model, as pointed out by Garti and Shelah in [10], witnesses the consistency of $\mathfrak{u}(\kappa) = \kappa^+ < 2^\kappa$ (Here $\mathfrak{u}(\kappa) = \min\{|\mathcal{B}|: \mathcal{B}$ is a base for a uniform ultrafilter on $\kappa\}$). See also [4].

Here we present a modification of the forcing construction introduced by Džamonja and Shelah in [6], which allows us to prove that if κ is a supercompact cardinal and $\kappa < \kappa^*$ with κ^* regular, then there is a generic extension of the universe in which cardinals have not been changed and $\mathfrak{u}(\kappa) = \kappa^*$. As in [6], we use a long iteration of forcings each with an initial choice of ultrafilter, and truncate the iteration at an

* Corresponding author.
E-mail address: vera.fischer@univie.ac.at (V. Fischer).

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appropriate point of the desired cofinality (for us, κ^*). Specifically, we truncate at a limit of stages at which the chosen ultrafilter is the canonical one given by an elementary embedding j: namely, those X such that $\kappa \in j(X)$. The proof in [6] that such a stage exists is by a careful, bottom-up recursion. We take a more top–down approach (for which see also [8]) that freely refers to the canonical ultrafilter in the eventual full generic extension, of which the desired ultrafilters are just restrictions. We believe this perspective makes for a more streamlined proof. As in [6], we may in addition ensure that each of the restricted ultrafilters contains a Mathias generic for its smaller restrictions, yielding an ultrafilter generated by these κ^*-many Mathias generics and so obtaining $\mathfrak{u}(\kappa) = \kappa^*$.

Moreover our construction allows us to decide the values of many of the higher analogues of the known classical cardinal characteristics of the continuum, as we can interleave arbitrary κ-directed closed posets cofinally in the iteration. The detailed construction of our model is presented in Section 3, while our applications appear in Section 4.

Thus our main result states the following:

Theorem 1. *Suppose κ is a supercompact cardinal, κ^* is a regular cardinal with $\kappa < \kappa^* \leq \Gamma$ and Γ satisfies $\Gamma^\kappa = \Gamma$. Then there is forcing extension in which cardinals have not been changed satisfying:*

$$\kappa^* = \mathfrak{u}(\kappa) = \mathfrak{b}(\kappa) = \mathfrak{d}(\kappa) = \mathfrak{a}(\kappa) = \mathfrak{s}(\kappa) = \mathfrak{r}(\kappa) = \mathrm{cov}(\mathcal{M}_\kappa)$$
$$= \mathrm{add}(\mathcal{M}_\kappa) = \mathrm{non}(\mathcal{M}_\kappa) = \mathrm{cof}(\mathcal{M}_\kappa) \text{ and } 2^\kappa = \Gamma.$$

If in addition $(\Gamma)^{<\kappa^*} \leq \Gamma$ then we can also provide that $\mathfrak{p}(\kappa) = \mathfrak{t}(\kappa) = \mathfrak{h}_{\mathcal{W}}(\kappa) = \kappa^*$ where $\mathcal{W}$ is a κ-complete ultrafilter on κ.

We also establish some of the natural inequalities between the generalized characteristics which in the countable case are well known.

2. Preliminaries

Let κ be a supercompact cardinal. Recall that this means that for all $\lambda \geq \kappa$ there is an elementary embedding $j : V \to M$ with critical point κ, $j(\kappa) > \lambda$ and $M^\lambda \subseteq M$.

One of the main properties of supercompact cardinals that will be used throughout the paper is the existence of the well-known Laver preparation, which makes the supercompactness of κ indestructible by subsequent forcing with κ-directed-closed partial orders.

Theorem 2. *(Laver, [13]) If κ is supercompact, then there exists a κ-cc partial ordering S_κ of size κ such that in V^{S_κ}, κ is supercompact and remains supercompact after forcing with any κ-directed closed partial order.*

The main lemma used to obtain this theorem is the statement that for any supercompact cardinal κ there exists a *Laver diamond*. That is, there is a function $h : \kappa \to V_\kappa$ such that for every set x and every cardinal λ, there is an elementary embedding $j : V \to M$ with critical point κ, $j(\kappa) > \lambda$, $M^\lambda \subseteq M$ and $j(h)(\kappa) = x$.

Given such a function, the Laver preparation S_κ is given explicitly as a reverse Easton iteration $(S_\alpha, \dot{R}_\beta : \alpha \leq \kappa, \beta < \kappa)$, defined alongside a sequence of cardinals $(\lambda_\alpha : \alpha < \kappa)$ by induction on $\alpha < \kappa$ as follows.

- If α is a cardinal and $h(\alpha) = (\dot{P}, \lambda)$, where λ is a cardinal, $\dot{P}$ is an S_α name for a $< \alpha$-directed closed forcing, and for all $\beta < \alpha$, $\lambda_\beta < \alpha$, we let $\dot{R}_\alpha := \dot{P}$ and $\lambda_\alpha = \lambda$.
- Otherwise, we let $\dot{R}_\alpha$ be the canonical name for the trivial forcing and $\lambda_\alpha = \sup_{\beta<\alpha} \lambda_\beta$.