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# Bounded stationary reflection II

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Bounded stationary reflection at a cardinal $\lambda$ is the assertion that every stationary subset of $\lambda$ reflects but there is a stationary subset of $\lambda$ that does not reflect at arbitrarily high cofinalities. We produce a variety of models in which bounded stationary reflection holds. These include models in which bounded stationary reflection holds at the successor of every singular cardinal $\mu > \aleph_\omega$ and models in which bounded stationary reflection holds at $\mu^+$ but the approachability property fails at $\mu$.



## 1. Introduction

The reflection of stationary sets is a topic of fundamental interest in the study of combinatorial set theory, large cardinals, and inner model theory and provides a useful tool for the investigation of the tension between compactness and incompactness phenomena. In this paper, we extend results, inspired by a question of Eisworth, of Cummings and the author [3]. We start by reviewing the relevant definitions and providing an outline of the structure of the paper.

**Definition 1.1.** Let $\lambda > \omega_1$ be a regular cardinal.

1. If $S \subseteq \lambda$ is a stationary set and $\alpha < \lambda$ has uncountable cofinality, then $S$ *reflects at* $\alpha$ if $S \cap \alpha$ is stationary in $\alpha$. $S$ *reflects* if there is $\alpha < \lambda$ with uncountable cofinality such that $S$ reflects at $\alpha$.
2. If $\mu$ is a singular cardinal and $\lambda = \mu^+$, then $\mathrm{Refl}(\lambda)$ holds if every stationary subset of $\lambda$ reflects.

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3. If $\mu$ is a singular cardinal, $\lambda = \mu^+$, and $S \subseteq \lambda$ is stationary, then $S$ *reflects at arbitrarily high cofinalities* if, for all $\kappa < \mu$, there is $\alpha < \lambda$ such that $\mathrm{cf}(\alpha) \geq \kappa$ and $S$ reflects at $\alpha$.
4. If $\mu$ is a singular cardinal and $\lambda = \mu^+$, then $\mathrm{bRefl}(\lambda)$ (bounded stationary reflection at $\lambda$) holds if $\mathrm{Refl}(\lambda)$ holds but there is a stationary $T \subseteq \lambda$ that does not reflect at arbitrarily high cofinalities.

Eisworth [4] asked whether $\mathrm{bRefl}(\lambda)$ is consistent when $\lambda$ is the successor of a singular cardinal. $\mathrm{bRefl}(\aleph_{\omega+1})$ is easily seen to be inconsistent, but Cummings and the author showed in [3] that, for other values of $\lambda$, $\mathrm{bRefl}(\lambda)$ is consistent modulo large cardinal assumptions. In particular, the following theorem was proven.

**Theorem 1.2.** *Suppose there is a proper class of supercompact cardinals. Then there is a class forcing extension in which, for every singular cardinal $\mu > \aleph_\omega$ such that $\mu$ is not a cardinal fixed point,* $\mathrm{bRefl}(\mu^+)$ *holds.*

This left open the question of whether it is consistent that $\mathrm{bRefl}(\mu^+)$ holds for every singular cardinal $\mu > \aleph_\omega$. In this paper, we answer this question affirmatively and prove a number of variations on Theorem 1.2.

In Section 2, we briefly discuss the notion of approachability before defining some of the forcing posets to be used throughout the paper and introducing their basic properties. In Section 3, we prove a general lemma about iteratively destroying stationary sets. In Section 4, we prove a dense version of Theorem 1.2 by producing a model in which $\mathrm{Refl}(\aleph_{\omega\cdot 2+1})$ holds and, for every stationary $S \subseteq S^{\aleph_{\omega\cdot 2+1}}_{<\aleph_\omega}$, there is a stationary $T \subseteq S$ that does not reflect at arbitrarily high cofinalities. In Section 5, we prove a global version of Theorem 1.2 by producing a model in which, for every singular cardinal $\mu > \aleph_\omega$, $\mathrm{bRefl}(\mu^+)$ holds.

The proofs of the results in Sections 4 and 5 and in [3] rely heavily on the approachability property holding in the final model. The relationship between approachability and stationary reflection is complicated and interesting, and in the last two sections of this paper we investigate the extent to which we can get bounded stationary reflection together with the failure of approachability. In Section 6, we produce a model with a singular cardinal $\mu$ such that $AP_\mu$ fails and $\mathrm{bRefl}(\mu^+)$ holds. In this model $\mu$ is a limit of cardinals which are supercompact in an outer model. In Section 7, we show that this result can be attained with $\mu = \aleph_{\omega^2\cdot 2}$.

Our notation is for the most part standard. The primary reference for all undefined notions and notations is [7]. If $\kappa < \lambda$ are infinite cardinals, with $\kappa$ regular, then $S^\lambda_\kappa = \{\alpha < \lambda \mid \mathrm{cf}(\alpha) = \kappa\}$. Expressions such as $S^\lambda_{>\kappa}$ or $S^\lambda_{\geq\kappa}$ are defined in the obvious way. If $X$ is a set of ordinals, then $\mathrm{nacc}(X)$ (the set of non-accumulation points of $X$) is the set $\{\alpha \in X \mid \sup(X \cap \alpha) < \alpha\}$, $X'$ is the set of limit points of $X$ (i.e. $X \setminus \mathrm{nacc}(X)$), and $\mathrm{otp}(X)$ is the order type of $X$. If $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\zeta \mid \xi \leq \gamma, \zeta < \gamma\rangle$ is a forcing iteration with supports of size $\mu$ for some cardinal $\mu$, we will frequently write $\Vdash_\xi$ instead of $\Vdash_{\mathbb{P}_\xi}$. Conditions of $\mathbb{P}_\gamma$ are thought of as functions $p$ such that $\mathrm{dom}(p) \in [\gamma]^{\leq\mu}$ and, for all $\zeta \in \mathrm{dom}(p)$, $\Vdash_\zeta$ "$p(\zeta) \in \dot{\mathbb{Q}}_\zeta$." For $\zeta < \xi \leq \gamma$, we let $\dot{\mathbb{P}}_{\zeta,\xi}$ be a $\mathbb{P}_\zeta$-name such that $\mathbb{P}_\xi \cong \mathbb{P}_\zeta * \dot{\mathbb{P}}_{\zeta,\xi}$.

## 2. Approachability and forcing preliminaries

**Definition 2.1.** Let $\lambda$ be a regular, uncountable cardinal.

1. Let $\vec{a} = \langle a_\alpha \mid \alpha < \lambda\rangle$ be a sequence of bounded subsets of $\lambda$. If $\gamma < \lambda$, $\gamma$ is *approachable with respect to* $\vec{a}$ if there is an unbounded $A \subseteq \gamma$ such that $\mathrm{otp}(A) = \mathrm{cf}(\gamma)$ and, for every $\beta < \gamma$, there is $\alpha < \gamma$ such that $A \cap \beta = a_\alpha$.
2. If $B \subseteq \lambda$, then $B \in I[\lambda]$ if there is a club $C \subseteq \lambda$ and a sequence $\vec{a} = \langle a_\alpha \mid \alpha < \lambda\rangle$ of bounded subsets of $\lambda$ such that, for every $\gamma \in B \cap C$, $\mathrm{cf}(\gamma) < \gamma$ and $\gamma$ is approachable with respect to $\vec{a}$.
3. If $\mu$ is a singular cardinal and $\lambda = \mu^+$, then $AP_\mu$ is the assertion that $\lambda \in I[\lambda]$.