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Vaught's conjecture for quite o-minimal theories

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ABSTRACT

We study Vaught's problem for quite o-minimal theories. Quite o-minimal theories form a subclass of the class of weakly o-minimal theories preserving a series of properties of o-minimal theories. We investigate quite o-minimal theories having fewer than 2^ω countable models and prove that the Exchange Principle for algebraic closure holds in any model of such a theory and also we prove binarity of these theories. The main result of the paper is that any quite o-minimal theory has either 2^ω countable models or 6^a3^b countable models, where a and b are natural numbers.

1. Preliminaries

Let L be a countable first-order language. Throughout the paper we consider L-structures and their complete elementary theories, and assume that L contains a symbol of binary relation $<$, which is interpreted as a linear order in these structures. An *open interval* in such a structure M is a parametrically definable subset of M of the form $I = \{c \in M : M \models a < c < b\}$ for some $a, b \in M \cup \{-\infty, \infty\}$ with $a < b$. Similarly, we may define *closed, half open-half closed*, etc., *intervals* in M. An arbitrary point $a \in M$ we can also represent as an interval $[a, a]$. By an *interval* in M we shall mean, ambiguously, any of the above types

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of intervals in M. A subset A of a linearly ordered structure M is *convex* if for any $a, b \in A$ and $c \in M$ whenever $a < c < b$ we have $c \in A$.

The present work deals with the notion of *weak o-minimality*, which initially deeply studied by D. Macpherson, D. Marker, and C. Steinhorn in [11]. A *weakly o-minimal structure* is a linearly ordered structure $M = \langle M, =, <, \ldots \rangle$ such that any definable (with parameters) subset of the structure M is a finite union of convex sets in M. We recall that such a structure M is said to be *o-minimal* if any definable (with parameters) subset of M is the union of finitely many intervals and points in M. Thus, the weak o-minimality generalizes the notion of o-minimality. Real closed fields with a proper convex valuation ring provide an important example of weakly o-minimal (not o-minimal) structures.

Let A, B be arbitrary subsets of a linearly ordered structure M. Then the expression $A < B$ means that $a < b$ whenever $a \in A$ and $b \in B$. The expression $A < b$ (respectively, $b < A$) means that $A < \{b\}$ ($\{b\} < A$). We denote by A^+ (respectively, A^-) the set of elements $b \in M$ with $A < b$ ($b < A$). For an arbitrary type p we denote by $p(M)$ the set of realizations of p in M. For an arbitrary tuple $\bar{b} = \langle b_1, b_2, \ldots, b_n \rangle$ of length n we denote by $\bar{b}_i$ the tuple $\langle b_1, b_2, \ldots, b_i \rangle$ for any $1 \le i \le n-1$. If $B \subseteq M$ and E is an equivalence relation on B then we denote by B/E the set of E-classes (lying in B). If f is a function on M then we denote the domain of f by $\mathrm{Dom}(f)$ and its range by $\mathrm{Range}(f)$. A theory T is *binary* if any formula of T is equivalent in T to a Boolean combination of formulas with at most two free variables.

In the following definitions we assume that M is a weakly o-minimal structure, $A, B \subseteq M$, M is $|A|^+$-saturated, and $p, q \in S_1(A)$ are non-algebraic types.

Definition 1.1. (B.S. Baizhanov, [3]) We say that p is not *weakly orthogonal* to q ($p \not\perp^w q$) if there are an A-definable formula $H(x, y)$, $a \in p(M)$, and $b_1, b_2 \in q(M)$ such that $b_1 \in H(M, a)$ and $b_2 \notin H(M, a)$.

In other words, p is *weakly orthogonal* to q ($p \perp^w q$) if $p(x) \cup q(y)$ has a unique extension to a complete 2-type over A.

Lemma 1.2. ([3, Corollary 34 (iii)]) *The relation $\not\perp^w$ of the weak non-orthogonality is an equivalence relation on $S_1(A)$.*

In [7], quite o-minimal theories were introduced forming a subclass of the class of weakly o-minimal theories and preserving a series of properties for o-minimal theories. For instance, in [10], $\aleph_0$-categorical quite o-minimal theories were completely described. This description implies their binarity (the similar result holds for $\aleph_0$-categorical o-minimal theories).

Definition 1.3. [7] We say that p is not *quite orthogonal* to q ($p \not\perp^q q$) if there is an A-definable bijection $f : p(M) \to q(M)$. We say that a weakly o-minimal theory is *quite o-minimal* if the relations of weak and quite orthogonality coincide for 1-types over arbitrary sets of models of the given theory.

Lemma 1.4. *Any o-minimal theory is quite o-minimal.*

Proof of Lemma 1.4. Let T be an o-minimal theory, $M \models T$, $A \subseteq M$, M be $|A|^+$-saturated, $p, q \in S_1(A)$ be non-algebraic. We assume that $p \not\perp^w q$. Then there are an A-definable formula $H(x, y)$, $a \in p(M)$, $b_1, b_2 \in q(M)$ such that $b_1 \in H(M, a)$ and $b_2 \notin H(M, a)$. By o-minimality, $H(M, a)$ is a union of finitely many intervals and points. Without loss of generality we assume that $H(M, a)$ is convex and $b_1 < b_2$. Then there is $b \in q(M)$ such that $b_1 < b < b_2$ and b is an endpoint of $H(M, a)$, hence $b \in \mathrm{dcl}(A \cup \{a\})$. Thus, there is an A-definable function f such that $f(a) = b$ and f is a bijection of $p(M)$ onto $q(M)$. $\square$

The ordered field of real algebraic numbers expanded by a unary predicate $(-\alpha, \alpha)$, where α is an arbitrary real transcendental number (considered in [11]), provides an important example of quite o-minimal theories.