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Pseudo real closed fields, pseudo p-adically closed fields and NTP_2

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ABSTRACT

The main result of this paper is a positive answer to the Conjecture 5.1 of [14] by A. Chernikov, I. Kaplan and P. Simon: If M is a PRC field, then $Th(M)$ is NTP_2 if and only if M is bounded. In the case of PpC fields, we prove that if M is a bounded PpC field, then $Th(M)$ is NTP_2. We also generalize this result to obtain that, if M is a bounded PRC or PpC field with exactly n orders or p-adic valuations respectively, then $Th(M)$ is strong of burden n. This also allows us to explicitly compute the burden of types, and to describe forking.

1. Introduction

A *pseudo algebraically closed field* (PAC field) is a field M such that every absolutely irreducible affine variety defined over M has an M-rational point. The concept of a PAC field was introduced by J. Ax in [2] and has been extensively studied. The above definition of PAC field has an equivalent model-theoretic version: M is existentially closed (in the language of rings) in each regular field extension of M.

The notion of PAC field has been generalized by S. Basarab in [3] and then by A. Prestel in [35] to ordered fields. Prestel calls a field M *pseudo real closed* (PRC) if M is existentially closed (in the language of rings) in each regular field extension N to which all orderings of M extend. PRC fields were extensively studied by L. van den Dries in [42], Prestel in [35], M. Jarden in [25–27], Basarab in [5] and [4], and others.

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In analogy to PRC fields, C. Grob [19], Jarden and D. Haran [20] studied the class of pseudo p-adically closed fields. A field M is called a *pseudo p-adically closed* (PpC) if M is existentially closed (in the language of rings) in each regular field extension N to which all the p-adic valuations of M can be extended by p-adic valuations on N. PpC fields have also been studied by I. Efrat and Jarden in [18], Jarden in [28] and others.

The class of PRC fields contains strictly the classes of PAC fields of characteristic 0 and real closed fields (RCF) and the class of PpC fields contains the p-adically closed fields (pCF). It is known that the theories RCF and pCF are NIP. Duret showed in [16] that the complete theory of a PAC field which is not separably closed is not NIP. In [9] Z. Chatzidakis and A. Pillay proved that if M is a bounded (i.e. for any $n \in \mathbb{N}$, M has only finitely many extensions of degree n) PAC field, then $Th(M)$ is simple. In [7] Chatzidakis proved that if M is a PAC field and $Th(M)$ is simple, then M is bounded. PRC and PpC fields were extensively studied, but mainly from the perspective of algebra (description of absolute Galois group, etc), elementary equivalence, decidability, etc. Their stability theoretic properties had not been studied. In this paper we study the stability theoretic properties of the classes of PRC and PpC fields.

In Theorem 4.10 we generalize the result of Duret and we show that the complete theory of a PRC field which is neither algebraically closed nor real closed is not NIP. In Corollary 7.4 we show that the complete theory of a bounded PpC which is not p-adically closed is not NIP. The general case for PpC is still in progress, the main obstacle is that the p-adic valuations are not necessarily definable, and that algebraic extensions are not necessarily PpC.

The class of NTP$_2$ theories (theories without the tree property of the second kind, see Definition 4.1) was defined by Shelah in [39] in the 1980's and contains strictly the classes of simple and NIP theories. Recently the class of NTP$_2$ theories has been particularly studied and contains new important examples, A. Chernikov showed in [10] that any ultra-product of p-adics is NTP$_2$. A. Chernikov and M. Hils showed in [12] that a σ-Henselian valued difference field of equicharacteristic 0 is NTP$_2$, provided both the residue difference field and the value group (as an ordered difference group) are NTP$_2$. There are not many more examples of strictly NTP_2 theories. A. Chernikov, I. Kaplan and P. Simon conjectured in [14, Conjecture 5.1] that if M is a PRC field then $Th(M)$ is NTP$_2$ if and only M is bounded. Similarly if M is a PpC field.

The main result of this paper is a positive answer to the conjecture by Chernikov, Kaplan and Simon for the case of PRC fields (Theorem 4.23). In fact for bounded PRC fields we obtain a stronger result: In Theorem 4.22 we show that if M is a bounded PRC field with exactly n orders, then $Th(M)$ is *strong* of burden n. We also show that $Th(M)$ is not rosy (Corollary 4.20) and resilient (Theorem 4.30). The class of resilient theories contains the class of NIP theories and is contained in the class of NTP$_2$ theories. It is an open question to know whether NTP$_2$ implies resilience.

The case of PpC fields is more delicate, and we obtain only one direction of the conjecture. In Theorem 8.5 we show that the theory of a bounded PpC field with exactly n p-adic valuations is strong of burden n and in Theorem 8.5 we show that this theory is also resilient. That unbounded PpC fields have TP_2 will be discussed in another paper. The problem arises again from the fact that an algebraic extension of a PpC field is not necessarily PpC.

Independently, W. Johnson [29] has shown that the model companion of the theory of fields with several independent orderings has NTP$_2$, as well as characterized forking, extension bases, the burden, and several other results. He also obtains similar results for the class of existentially closed fields with several valuations, or with several p-adic valuations. Some of his results follow from ours, since his fields are bounded, and PRC or PpC in case of several orderings or p-adic valuations. His results on fields with several valuations however cannot be obtained by our methods.

The organization of the paper is as follows: In section 2 we give the required preliminaries on ordered fields and pseudo real closed fields. In section 3 we study the theory of bounded PRC fields from a model theoretic point of view. We work in a fixed complete theory of a bounded PRC field, and we enrich the