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A generalization of the Łoś–Tarski preservation theorem

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A B S T R A C T

We present new parameterized preservation properties that provide for each natural number k, semantic characterizations of the $\exists^k\forall^*$ and $\forall^k\exists^*$ prefix classes of first order logic sentences, over the class of all structures and for arbitrary finite vocabularies. These properties, that we call *preservation under substructures modulo k-cruxes* and *preservation under k-ary covered extensions* respectively, correspond exactly to the classical properties of preservation under substructures and preservation under extensions, when k equals 0. As a consequence, we get a parameterized generalization of the Łoś–Tarski preservation theorem for sentences, in both its substructural and extensional forms. We call our characterizations collectively the *generalized Łoś–Tarski theorem for sentences*. We generalize this theorem to theories, by showing that theories that are preserved under k-ary covered extensions are characterized by theories of $\forall^k\exists^*$ sentences, and theories that are preserved under substructures modulo k-cruxes, are equivalent, under a well-motivated model-theoretic hypothesis, to theories of $\exists^k\forall^*$ sentences. In contrast to existing preservation properties in the literature that characterize the Σ^0_2 and Π^0_2 prefix classes of FO sentences, our preservation properties are combinatorial and finitary in nature, and stay non-trivial over finite structures as well.

1. Introduction

Preservation theorems in first order logic (henceforth abbreviated FO) have been extensively studied in model theory. An FO preservation theorem for a model-theoretic operation syntactically characterizes elementary classes of structures that are closed under that operation. A classical preservation theorem (also one of the earliest) is the Łoś–Tarski theorem, which states that over the class of all (arbitrary) structures, an FO sentence is preserved under substructures if, and only if, it is equivalent to a universal sentence (see Theorem 3.2.2 in [3]). In dual form, the theorem states that an FO sentence is preserved under extensions if, and only if, it is equivalent to an existential sentence. It is well known that if the vocabulary contains

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only relation symbols, then the sizes of the minimal models of a sentence preserved under extensions are no larger than the number of quantifiers in any equivalent existential sentence. Thus, the dual form of the Łoś–Tarski theorem not only asserts the equivalence of a syntactic and a semantic class of FO sentences, but also yields a relation between a quantitative model-theoretic property (i.e., sizes of minimal models) of a sentence in the semantic class and the count of quantifiers in an equivalent sentence in the syntactic class.

A syntactic subclass of FO that is semantically richer than the universal and existential classes of sentences, is the Σ_2^0 class – the class of all prenex sentences having prefix structure of the form $\exists^*\forall^*$, i.e. sentences whose prefix structure consists of at most two blocks of quantifiers, with the leading block being existential. The literature contains several semantic characterizations, over the class of all structures, for this syntactic class using preservation properties defined in terms of notions such as ascending chains, descending chains, and 1-sandwiches (see Theorem 3.2.3, Proposition 5.2.16 and Theorem 5.2.6 in [3]). These results, in dual form, give semantic characterizations of the Π_2^0 class, which is the class of all $\forall^*\exists^*$ sentences, i.e. prenex sentences whose prefix contains at most two blocks of quantifiers with the leading block being universal. However, none of these characterizations relates quantifier counts in the aforementioned syntactic classes to any model-theoretic properties.

In this paper, we take a step towards addressing this problem. Specifically, we present new preservation theorems that provide semantic characterizations of sentences in prenex normal form, having quantifier prefixes of the form $\exists^k\forall^*$ or $\forall^k\exists^*$, i.e., quantifier prefixes consisting of at most two blocks of quantifiers and in which the leading block has k quantifiers for a given natural number k. Towards these theorems, we introduce, for a given sentence φ and a model $\mathfrak{A}$ of φ, the notions of *a k-crux of $\mathfrak{A}$ with respect to φ* and *a substructure of $\mathfrak{A}$ modulo a k-crux*. The latter notion corresponds exactly to the classical notion of substructure when k is equal to 0. We define the property of *preservation under substructures modulo k-cruxes* as a natural parameterized generalization of the property of preservation under substructures. Likewise, on the dual front, we introduce the notions of *k-ary covers*, *k-ary covered extensions* and *preservation under k-ary covered extensions*. The latter two notions reduce to the classical notions of extension and preservation under extensions respectively, when k equals 0. Our preservation theorems give syntactic characterizations of the above preservation properties. Specifically, we show for every natural number k, that (i) an FO sentence is preserved under substructures modulo k-cruxes if, and only if, it is equivalent to a prenex sentence having quantifier prefix of the form $\exists^k\forall^*$, and (ii) an FO sentence is preserved under k-ary covered extensions if, and only if, it is equivalent to a prenex sentence having quantifier prefix of the form $\forall^k\exists^*$. To the best of our knowledge, these results, that we collectively call the *generalized Łoś–Tarski theorem for sentences*, are the first to relate natural quantitative properties of models of sentences in a semantic class to counts of leading quantifiers in equivalent $\exists^*\forall^*$ or $\forall^*\exists^*$ sentences. They provide new and finer characterizations of the Σ_2^0 and Π_2^0 prefix classes vis-à-vis the characterizations of these classes in the literature.

In contrast to the existing preservation properties alluded to earlier, that characterize the Σ_2^0 and Π_2^0 classes, our preservation properties are combinatorial and finitary in nature, and stay non-trivial over finite structures as well. There has been a recent renewal of interest in preservation theorems in the context of finite model theory. Since most preservation theorems fail[1] over the class of all finite structures, recent research [1,2,4,6,7] has focused attention on studying classical preservation theorems over 'well-behaved' classes of finite structures. In particular, Atserias, Dawar and Grohe showed in [2] that under suitable closure assumptions, classes of structures that are acyclic or of bounded degree admit the Łoś–Tarski theorem for sentences. They also show the Łoś–Tarski theorem for sentences to be true over the class of all structures of tree-width at most k, for each natural number k (though the theorem is not necessarily true over proper subclasses of these classes). In a recent work [18], we identified many interesting classes of finite structures that admit the generalized Łoś–Tarski theorem for sentences. Specific examples include the classes of words, trees (as partial orders), structures of bounded tree-depth, grids of bounded dimension, various well-known subclasses

[1] A notable exception is the homomorphism preservation theorem [12].