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Transductions in arithmetic

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This paper is dedicated to the memory of Franco Montagna

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ABSTRACT

In this paper we study a new relation between sentences: *transducibility*. The idea of transducibility is based on an analysis of Feferman's Theorem that the inconsistency of a theory U is interpretable over U. Transducibility is based on a converse of Feferman's Theorem: if a sentence is interpretable over a theory U, it is, in a sense that we will explain, an inconsistency statement for U over U.

We show that, for a wide class of theories U, transducibility coincides with interpretability over U and, for an even wider class, it coincides with Π_1-conservativity over U. Thus, transducibility provides a new way of looking at interpretability and Π_1-conservativity. On the other hand, we will show that transducibility admits variations that are distinct from interpretability and Π_1-conservativity.

We show that transducibility satisfies the interpretability logic ILM.

1. Introduction

In this paper we provide new characterizations of interpretability for essentially reflexive theories and of Π_1-conservativity for theories extending Elementary Arithmetic EA (aka $\mathrm{I}\Delta_0+\mathsf{Exp}$). These characterizations stand in the tradition of characterizations such as the Orey-Hájek Characterization and the Friedman Characterization, but they are of a different flavor. Our approach uncovers a connection between interpretability and Π_1-conservativity, on the one hand, and inconsistency statements of provability predicates satisfying the Hilbert–Bernays–Löb conditions, on the other.

We can view what is achieved in the paper from various other perspectives. The paper is a study of *role provability predicates* as is explained in Subsection 1.1. It provides a converse of a beautiful theorem due

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to Feferman as is explained in Subsection 1.2. It provides a rather general arithmetical semantics for the interpretability logic ILM. See Subsection 1.3.

The present paper is closely related to two of my other papers. The first is [57], which is to appear in the Bulletin of Symbolic Logic. The second is a somewhat more philosophical paper [59]. The present paper can be read independently of its two companions.

1.1. Role provability predicates

Syntactical approaches to modality come in two flavors. A first idea is to add a predicate or predicates to a language that has sufficient coding possibilities. Then, we stipulate that the predicate, considered as a predicate of sentences, satisfies a number of desired modal properties. An important question is which properties we can consistently (or also conservatively) demand of such predicates and whether we can define a Kripke style semantics for them. For examples of this approach, see e.g. [33,39,49,43,25,22,47].

A second approach is the modal study of predicates that are definable in theories with sufficient coding possibilities. This line of research usually zooms in on specific predicates like *provability* and *interpretability*. Provability Logic is a perfect example of this study. The classical papers in this field are [17,34,37,46]. For expository texts, see: [9,8,35,29,48,1,23]. There are many variations.

1. Over EA, also known as $\mathrm{I}\Delta_0 + \mathsf{Exp}$, cutfree provability and ordinary provability are not equivalent. On the other hand they both validate Löb's Logic. See [51] and [32].
2. Over PA we can consider the predicates 'provable in PA with an oracle for Π_{n+1}-truth'. The logic of the hierarchy of such predicates is Japaridze's Logic GLP. See [28]. See also [8]. This logic was used by Lev Beklemishev to extract proof theoretic ordinals from its closed fragment. See e.g. [2,3,5,4].
3. We consider over the theory ZF, the predicate *truth in all transitive models of* ZF. This example was studied by Solovay in [46]. See also [8]. A closely related example is to consider truth in all V_κ where κ is inaccessible. See [8].
4. Let PA^2 be the first-order version of second order arithmetic. We may consider the arithmetization of provability in PA^2 with the ω-rule. This predicate was studied e.g. in [8].
5. Per Lindström studied Parikh provability in his paper [36].
6. Sy Friedman, Michael Rathjen and Andreas Weiermann study *slow provability* for PA in their paper [14]. The modal behavior of slow provability predicates is currently studied by Fedor Pakhomov and Paula Henk.
7. Over EA, provability with an oracle for Σ_1-truth and ordinary provability are not equivalent. On the other hand they both validate Löb's Logic. See [60].
8. Graham Leach-Krouse studied an internal version of Ω-validity over ZFC with the von Neumann interpretation.
9. A new kind of predicates called *supremum adapters* is studied by Paula Henk.

All predicates in the above list validate Löb's Logic. There are however other modally interesting predicates. The principal of the alternative unary predicates is the Feferman Predicate that was introduced in [12]. It was studied in [38,50,45]. Of a quite different kind is the binary predicate for interpretability over a given theory. This can be viewed as a generalization of ordinary provability. We refer the reader to e.g. [29,53,31,1,18]. An alternative arithmetical interpretation of interpretability logics is provided by various notions of conservativity. See [19]. In the present paper we will provide yet another interpretation: transducibility.

This paper is in the second tradition: we treat modal predicates as *objets trouvés*. They are already present in a given theory. On the other hand, we will not zoom in on specific predicates in the given theory, but we will be interested in the totality of predicates over the given theory satisfying such-and-such properties. The appropriate analogy is as follows. A predicate of a theory satisfying a given modal theory is like a model