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Separating club-guessing principles in the presence of fat forcing axioms ☆

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A B S T R A C T

We separate various weak forms of Club Guessing at ω_1 in the presence of $2^{\aleph_0}$ large, Martin's Axiom, and related forcing axioms.
We also answer a question of Abraham and Cummings concerning the consistency of the failure of a certain polychromatic Ramsey statement together with the continuum large.
All these models are generic extensions via finite support iterations with symmetric systems of structures as side conditions, possibly enhanced with ω-sequences of predicates, and in which the iterands are taken from a relatively small class of forcing notions.
We also prove that the natural forcing for adding a large symmetric system of structures (the first member in all our iterations) adds $\aleph_1$-many reals but preserves CH.

1. Introduction

One is sometimes faced with the problem of building a model of set theory satisfying the following two requirements.

(1) $2^{\aleph_0} > \aleph_2$ holds in the model.
(2) Some particular combinatorial principle P of the form "For all x there is some y such that $Q(x, y)$", where $Q(x, y)$ is sufficiently absolute, holds in the model. Furthermore, for each x there is a natural proper forcing adding a y such that $Q(x, y)$. Hence, P can be forced by means of a countable support iteration of proper forcings, but in the corresponding extension $2^{\aleph_0} \leq \aleph_2$ necessarily holds.

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The method of iterated forcing with finite supports and symmetric systems of submodels as side conditions was developed in [6] in order to resolve the tension between (1) and (2) in various situations (see [6] and [7] for background information). Variants of this method have been subsequently investigated in [7] and [3].

One of the central themes of the present article is the construction of iterations as in [6] where the iterands are chosen from some relatively small class of posets (let us call these constructions of the first type). The other central theme is a new variation of the general method from [6] obtained from associating sequences of predicates with the submodels in the side conditions of the iteration (this gives rise to the constructions of the second type). The main focus in this article is the separation of club-guessing principles at ω_1 in the presence of forcing axioms for relatively small classes of posets but with respect to large collections of dense sets. The corresponding models are obtained as forcing extensions via constructions of either the first or the second type. Also, using a construction of the first type we answer a question of Abraham–Cummings in the context of polychromatic Ramsey theory [1].

The rest of the paper is structured as follows. In the next subsection we prove several implications and non-implications between weak forms of Club Guessing at ω_1, and present our main theorems (Theorems 1.15 and 1.16). In Section 2 we take a look at the forcing for adding a symmetric system of submodels by finite conditions. This forcing is either the first step or is subsumed in the first step in all our iterations. We show that this forcing adds $\aleph_1$-many reals but preserves CH. In Section 3 we prove Theorems 1.15 and 1.16. Finally, in Section 4 we deal with the Abraham–Cummings question. Most of our notation will be standard (see e.g. [12] or [15]) but we will also be introducing additional pieces of notation as we need them.

1.1. Weak forms of Club Guessing

A *ladder system* is a sequence $\langle C_\delta \mid \delta \in \mathrm{Lim}(\omega_1)\rangle$, where each C_δ is a club of δ of order type ω. *Club Guessing* (CG) is the well-known weakening of $\diamondsuit$ saying that there is a ladder system $\langle C_\delta \mid \delta \in \mathrm{Lim}(\omega_1)\rangle$ which guesses clubs C of ω_1, in the sense that for every such C there is some δ such that a tail of C_δ is contained in C. In this subsection we focus our attention on certain weakenings of CG. The web of implications between these principles will be immediate. We will then point out how to prove several non-implications between these principles, with a focus on what can be obtained in the presence of forcing axioms for large families of dense sets. Finally we present our main separation theorems, to be proved in Section 3.[1]

Kunen's Axiom (KA), also known as *Interval Hitting Principle* (see for example [10]), is the following statement first considered by Kunen: There is a ladder system $\langle C_\delta \mid \delta \in \mathrm{Lim}(\omega_1)\rangle$ with the property that for every club $C \subseteq \omega_1$ there is some δ such that $[C_\delta(n), C_\delta(n+1)) \cap C \neq \emptyset$ for co-finitely many $n < \omega$ (where, here and throughout the paper, $X(\xi)$ denotes the ξth member of X if X is a set of ordinals).

$\mho$ (mho), first defined by Todorčević [21], says that there is a sequence of continuous colorings $g_\delta : \delta \longrightarrow \omega$, for $\delta \in \mathrm{Lim}(\omega_1)$, where δ and ω are both endowed with the order topology (so the topology on ω is the discrete topology), such that for every club $C \subseteq \omega_1$ there is some δ with $g_\delta^{-1}(\{n\}) \cap C \neq \emptyset$ for all $n < \omega$.

It is clear that CG implies KA and that KA implies $\mho$.

Weak Club Guessing (WCG), first defined by Shelah [18], says that there is a ladder system $\langle C_\delta \mid \delta \in \mathrm{Lim}(\omega_1)\rangle$ such that every club of ω_1 has infinite intersection with some C_δ. Very Weak Club Guessing (VWCG), also first considered by Shelah, says that there is a set $\mathcal{X}$ of size $\aleph_1$ consisting of subsets of ω_1 of order type ω such that every club of ω_1 has infinite intersection with some member of $\mathcal{X}$.

One can weaken VWCG even further: Given a cardinal $\lambda \geq \aleph_1$, VWCG_λ says that there is a set $\mathcal{X}$ of size at most λ consisting of subsets of ω_1 of order type ω and such that every club of ω_1 has infinite intersection with some member of $\mathcal{X}$. $\neg\,\mathrm{VWCG}_\lambda$ is called $(*)^\omega_\lambda$ in [7] (Definition 1.10).

Obviously WCG implies VWCG and VWCG_λ implies VWCG_μ whenever $\aleph_1 \leq \lambda < \mu$.

[1] For further information on these (and other related) club-guessing principles, see for example [11,10], and [16].