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Undecidability through Fourier series

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ABSTRACT

In computability theory a variety of combinatorial systems are encountered (word problems, production systems) that exhibit undecidability properties. Here we seek such structures in the realm of Analysis, more specifically in the area of Fourier Analysis. The starting point is that sufficiently strongly convergent Fourier series give rise to predicates in the sense of first order predicate calculus by associating to any s-ary Fourier series the predicate "the Fourier coefficient with index $(n_1, \dots, n_s)$ is non-zero". We introduce production systems, viewed as counterparts of the combinatorial ones, that generate all recursively enumerable predicates in this way using as tools only elementary operations and functions from classical Analysis. The problem arises how simple such a system may be. It turns out that there is a connection between this question and an as yet unproved conjecture by R. Büchi. This is discussed in the second half of the paper.
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1. Introduction

The aim of the present note is to describe continuous structures from Analysis that exhibit undecidability properties comparable, to some extent, with the well known word problems from Logic and Algebra (e.g. [7], chapter 6). Here we seek such continuous structures in the area of Fourier Analysis which, thanks to its proximity to Algebra, lends itself as a natural candidate for this kind of questions.

The bridge from analysis to mathematical logic, as looked at in this paper, is established via the Fourier coefficients, the procedure being as follows. The universe of our objects of interest is the set $\mathscr{F}^+$ of all complex valued, 2π-periodic functions of several real variables that can be developed into a sufficiently strongly converging Fourier series $\sum_{n_1,\dots,n_s \in \mathbb{N}} a_{n_1 \dots n_s} e^{in_1\alpha_1} \cdots e^{in_s\alpha_s}$ (we refer to Section 2 for the precise setting). An s-ary first order predicate Γ is then defined by stipulating that $\Gamma(n_1, \dots, n_s)$ holds iff $a_{n_1 \dots n_s} \neq 0$.

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The question asked is what kind of predicates does one get if one constructs functions in $\mathscr{F}^+$ by the traditional methods of classical Analysis.

"Construction by traditional methods" shall mean that one begins with a subset $\mathscr{U} \subseteq \mathscr{F}^+$ consisting of some classical functions from Analysis such as the trigonometric functions, etc., that are considered "basic". Then one builds new functions out of $\mathscr{U}$ by applying a finite set of production rules $\mathscr{P} = \{P_1, \ldots, P_k\}$, where each P_i is a classical construction in Analysis such as integration, to take the derivative or to solve a differential equation.

With $\mathscr{U} \subseteq \mathscr{F}^+$ and $\mathscr{P}$ given, a function $F \in \mathscr{F}^+$ will be called *attainable* if it can be obtained by beginning with $\mathscr{U}$ and applying successively rules from $\mathscr{P}$ in a finite number of steps (Definition 3.2). The rules may be understood as procedures that operate on expressions (see Remark 3.15), thus, for functions, $\mathscr{U}$ and $\mathscr{P}$ play a role in some sense similar to that of generators and relations for groups.

Our interest focuses on the following three questions whose answers depend, of course, on the choice of the production system $\mathscr{U}; \mathscr{P}$:

(1) Are there attainable functions that exhibit a non-recursive behaviour?
(2) Are the attainable functions computable in the sense of Computable Analysis?
(3) How simple can a production system $\mathscr{U}; \mathscr{P}$ be if one wants to get a positive answer to (1)?

In this paper we contribute to (1) and (3), but restrict (2) to a simple remark for reasons of space. The results are stated in Sections 3 and 5. In order to handle the first question we rely on methods used earlier in [17–19,3]. For (3) we shall use an unexpected, albeit loose, connection between the classical Jacobi theta function ϑ_λ and a purely number theoretic problem of Büchi [2], also known as the five squares problem (see Section 5).

The problem of finding non-recursive properties in continuous structures of Analysis has found the interest of several authors. We mention A. Adler [1], J. Denef and L. Lipschitz [8], M. Pour-El and J. Richards [12, 13], D. Richardson [15] and in particular L. Rubel [16] (see also Chapter 9.2 in [20]) who constructs a hierarchy of real functions of increasing complexity based on a basic set of initial functions and a series of rules that include among other things the solving of differential equations. Further references may be found in Matiyasevich [10], chapter 9.

Interest in Analog Computation by continuous structures was also stimulated by the writings of Jack Copeland, see e.g. [5]. We also refer to the monograph on hypercomputation by A. Syropoulos [20].

2. Notation and preliminaries

In what follows $\mathscr{F}_s$ denotes the set of all continuous complex-valued functions that depend on s real arguments $\alpha_1, \ldots, \alpha_s$ that are 2π-periodic in each α_j and whose Fourier series expansion

$$F(\alpha_1, \ldots, \alpha_s) = \sum_{n_1, \ldots, n_s \in \mathbb{Z}} a_{n_1 \ldots n_s} e^{\mathrm{i} n_1 \alpha_1} \cdots e^{\mathrm{i} n_s \alpha_s} \tag{2.1}$$

satisfies

$$\sum_{n_1, \ldots, n_s \in \mathbb{Z}} |a_{n_1 \ldots n_s}| < \infty. \tag{2.1.a}$$

By $\mathscr{F}_s^+$ we denote the subset of all $F \in \mathscr{F}_s$ that satisfy the additional condition

$$a_{n_1 \ldots n_s} \geq 0; \text{ and } a_{n_1 \ldots n_s} = 0 \text{ if } n_j < 0 \text{ for some } j \leq s. \tag{2.1.b}$$

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