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# Almost structural completeness; an algebraic approach

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ABSTRACT

A deductive system is structurally complete if all of its admissible inference rules are derivable. For several important systems, like the modal logic S5, failure of structural completeness is caused only by the underivability of a passive rule, i.e., a rule whose premise is not unifiable by any substitution. Neglecting passive rules leads to the notion of almost structural completeness, that means, to the derivability of admissible non-passive rules. We investigate almost structural completeness for quasivarieties and varieties of general algebras. The results apply to all algebraizable deductive systems.

Firstly, various characterizations of almost structurally complete quasivarieties are presented. Two of them are general: the one expressed with finitely presented algebras, and the one expressed with subdirectly irreducible algebras. The next one is restricted to quasivarieties with the finite model property and equationally definable principal relative congruences, where the condition is verifiable on finite subdirectly irreducible algebras. Some connections with exact and projective unification are included.

Secondly, examples of almost structurally complete varieties are provided. Particular emphasis is put on varieties of closure algebras, that are known to constitute adequate semantics for normal extensions of the modal logic S4. A certain infinite family of such almost structurally complete, but not structurally complete, varieties is constructed. Every variety from this family has a finitely presented unifiable algebra which does not embed into any free algebra for this variety. Hence unification is not unitary there. This shows that almost structural completeness is strictly weaker than projective unification for varieties of closure algebras.



## 1. Introduction

The paper is a contribution to the study of admissible inference rules from an algebraic perspective. Roughly speaking, an inference rule is admissible for a deductive system (or a logic) $\mathcal{S}$ if it does not produce

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new theorems when added to $\mathcal{S}$ [43,45,46,48,49,66,78]. Clearly, every derivable rule is admissible but the converse does not need to hold. The concept of admissibility was formally introduced by Lorenzen in [55] but it appeared much earlier. A well known admissible rule is the *cut rule* of Gentzen's sequent systems for classical and intuitionistic logics. Note that for a given deductive system admissibility depends only on its theorems. Derivability does not share this property. In [41] a deductive system is constructed with the set of classical tautologies as the set of theorems but in which even the *Modus Ponens rule* is not derivable.

The study of admissibility for non-classical logics was stimulated by Friedman's problem. It asks whether the admissibility of rules is decidable for the intuitionistic logic [33]. This problem was solved by Rybakov in [78]. But admissibility was further investigated in intuitionistic and modal logics. New horizons were opened when Ghilardi successfully applied unification to admissibility problems. Ghilardi not only provided a new and elegant solution to Friedman's problem [34,35] but also gave new effective tools for studying admissibility of rules and linked logic and computer science.

In order to present the motivation for introducing almost structural completeness and for studying admissibility, we recall some notions from propositional logic. Let $\mathcal{L}$ be a propositional language, i.e., a set of logical connectives with ascribed arities, and let **Form** be the algebra of formulas in $\mathcal{L}$ over a denumerable set of variables. An (*inference*) *rule* is a pair from $\mathcal{P}_{fin}(Form) \times Form$, written as $\Phi/\varphi$, where $\mathcal{P}_{fin}(Form)$ is the set of all finite subsets of *Form*. By a *deductive system* we mean a pair $\mathcal{S} = (\mathbf{Form}, \vdash)$, where $\vdash$ is a (finitary structural) consequence relation, this is, a set of rules satisfying the following postulates[2] [19,30,31,70,72,75,83,84]: for all $\Phi, \Psi \in \mathcal{P}_{fin}(Form)$ and $\varphi \in Form$ we have

- if $\varphi \in \Phi$, then $\Phi \vdash \varphi$,
- if for all $\psi \in \Psi$, $\Phi \vdash \psi$, and $\Psi \vdash \varphi$, then $\Phi \vdash \varphi$,
- for every substitution $\sigma$ (i.e., an endomorphism of **Form**), if $\Phi \vdash \varphi$, then $\sigma(\Phi) \vdash \sigma(\varphi)$.

The set $\mathrm{Th}(\mathcal{S}) = \{\varphi \in Form \mid \emptyset \vdash \varphi\}$ is the set of *theorems* of $\mathcal{S}$.

A *logic* is a subset of *Form* which is closed under substitution and some *default* rules. For instance, for intermediate logics this is the *Modus Ponens rule* and for normal modal logics these are the *Modus Ponens* and the *Necessitation rules*. With a logic $L$ (with a set $R$ of default rules) one can associate the deductive system $\mathcal{S} = (\mathbf{Form}, \vdash)$ such that $L = \mathrm{Th}(\mathcal{S})$ and $\vdash$ is minimal with respect to $R \subseteq \vdash$.

A *basis* or an *axiomatization* of a deductive system $\mathcal{S}$ (or of a logic with a default set $R$ of rules) is a pair $(A, R)$ (or just a set $A$), where $A \subseteq \mathrm{Th}(\mathcal{S})$ and $R \subseteq \vdash$ are such that $\vdash$ is the smallest consequence relation containing $R \cup \{\emptyset/\alpha \mid \alpha \in A\}$. It means that $\Phi \vdash \varphi$ iff there is a *proof* or *derivation* (in some strict sense) from $A \cup \Phi$ for $\varphi$ by means of the rules from $R$.

Given a basis $(A, R)$ of a deductive system $\mathcal{S}$ one is interested whether a formula $\varphi$ is in $\mathrm{Th}(\mathcal{S})$, i.e., whether there is a proof of $\varphi$ from $A$ with the application of the rules from $R$. Thus the issue of the size of such proofs arises. Proofs of theorems may be shortened by allowing new rules. Such extension of $R$ may be done in two ways:

1. by adding derivable rules;
2. by adding admissible but non-derivable rules.

A rule is *derivable* if it is in $\vdash$. And a rule $\Phi/\varphi$ is *admissible* if for every substitution $\sigma$ we have $\sigma(\varphi) \in \mathrm{Th}(\mathcal{S})$ whenever $\sigma(\Phi) \subseteq \mathrm{Th}(\mathcal{S})$.

[2] We adopt the slightly modified definition from [31]. However it is also a common practice to use the term "deductive system" for the basis of deductive system in our sense.