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Canonical forking in AECs

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ABSTRACT

Boney and Grossberg [7] proved that every nice AEC has an independence relation. We prove that this relation is unique: in any given AEC, there can exist at most one independence relation that satisfies existence, extension, uniqueness and local character. While doing this, we study more generally the properties of independence relations for AECs and also prove a canonicity result for Shelah's good frames. The usual tools of first-order logic (like the finite equivalence relation theorem or the type amalgamation theorem in simple theories) are not available in this context. In addition to the loss of the compactness theorem, we have the added difficulty of not being able to assume that types are sets of formulas. We work axiomatically and develop new tools to understand this general framework.

1. Introduction

Let K be an abstract elementary class (AEC) which satisfies amalgamation, joint embedding, and which does not have maximal models. These assumptions allow us to work inside its monster model $\mathfrak{C}$. The main results of this paper are:

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(1) There is at most one independence relation satisfying existence, extension, uniqueness and local character (Corollary 5.19).
(2) Under some reasonable conditions, the coheir relation of [7] has local character and is canonical (Theorems 6.4 and 6.7).
(3) Shelah's weakly successful good λ-frames are canonical: an AEC can have at most one such frame (Theorem 6.13).

To understand the relevance of the results, some history is necessary.

In 1970, Shelah discovered the notion "$\text{tp}(\bar{a}/B)$ forks over A" (for $A \subseteq B$), a generalization of Morley's rank in ω-stable theories. Its basic properties were published in [20].

In 1974, Lascar [17, Theorem 4.9] established that for superstable theories, any relation between $\bar{a}$, B, A satisfying the basic properties of forking is Shelah's forking relation. In 1984, Harnik and Harrington [12, Theorem 5.8] extended Lascar's abstract characterization to stable theories. Their main device was the finite equivalence relation theorem. In 1997, Kim and Pillay [15, Theorem 4.2] published an extension to simple theories, using the independence theorem (also known as the type-amalgamation theorem).

This paper deals with the characterization of independence relations in various non-elementary classes. An early attempt on this problem can be found in Kolesnikov's [16], which focuses on some important particular cases (e.g. homogeneous model theory and classes of atomic models). We work in a more general context, and only rely on the abstract properties of independence. We cannot assume that types are sets of formulas, so work only with Galois (i.e. orbital) types.

In [21, Chapter II] (which later appeared as [26, Chapter V.B]), Shelah gave the first axiomatic definition of independence in AECs, and showed that it generalized first-order forking. In [25, Chapter II], Shelah gave a similar definition, localized to models of a particular size λ (the so-called "good λ-frame"). Shelah proved that a good frame existed, under very strong assumptions (typically, the class is required to be categorical in two consecutive cardinals).

Recently, working with a different set of assumptions (the existence of a monster model and tameness), Boney and Grossberg [7] gave conditions (namely a form of Galois stability and the extension property for coheir) under which an AEC has a global independence relation. This showed that one could study independence in a broad family of AECs. Our paper is strongly motivated by both [25, Chapter II] and [7].

The paper is structured as follows. In Section 2, we fix our notation, and review some of the basic concepts in the theory of AECs. In Section 3, we introduce independence relations, the main object of study of this paper, as well as some important properties they could satisfy, such as extension and uniqueness. We consider two examples: coheir and nonsplitting.

In Section 4, we prove a weaker version of (1) (Corollary 4.14) that has some extra assumptions. This is the core of the paper.

In Section 5, we go back to the properties listed in Section 3 and investigate relations between them. We show that some of the hypotheses in Corollary 4.14 are redundant. For example, we show that the symmetry and transitivity properties follow from existence, extension, uniqueness, and local character. We conclude by proving (1). Finally, in Section 6, we apply our methods to the coheir relation considered in [7] and to Shelah's good frames, proving (2) and (3).

While we work in a more general framework, the basic results of Sections 2–3 often have proofs that are very similar to their first-order analogs. Readers feeling confident in their knowledge of first-order nonforking can start reading directly from Section 4 and refer back to Sections 2–3 as needed.

This paper was written while the first and fourth authors were working on a Ph.D. under the direction of Rami Grossberg at Carnegie Mellon University. They would like to thank Professor Grossberg for his guidance and assistance in their research in general and in this work specifically.

An early version of this paper was circulated already in early 2014. Since that time, Theorem 5.14 has been used by the fourth author to build a good frame from amalgamation, tameness, and categoricity in