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Symmetry and the union of saturated models in superstable abstract elementary classes

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ABSTRACT

Our main result (Theorem 1) suggests a possible dividing line (μ-superstable + μ-symmetric) for abstract elementary classes without using extra set-theoretic assumptions or tameness. This theorem illuminates the structural side of such a dividing line.

Theorem 1. *Let $\mathcal{K}$ be an abstract elementary class with no maximal models of cardinality μ^+ which satisfies the joint embedding and amalgamation properties. Suppose $\mu \geq \mathrm{LS}(\mathcal{K})$. If $\mathcal{K}$ is μ- and μ^+-superstable and satisfies μ^+-symmetry, then for any increasing sequence $\langle M_i \in \mathcal{K}_{\geq\mu^+} \mid i < \theta < (\sup \|M_i\|)^+\rangle$ of μ^+-saturated models, $\bigcup_{i<\theta} M_i$ is μ^+-saturated.*

We also apply results of [18] and use towers to transfer symmetry from μ^+ down to μ in abstract elementary classes which are both μ- and μ^+-superstable:

Theorem 2. *Suppose $\mathcal{K}$ is an abstract elementary class satisfying the amalgamation and joint embedding properties and that $\mathcal{K}$ is both μ- and μ^+-superstable. If $\mathcal{K}$ has symmetry for non-μ^+-splitting, then $\mathcal{K}$ has symmetry for non-μ-splitting.*

In first-order logic, the statement, the union of any increasing sequence $\langle M_i \mid i < \theta\rangle$ of saturated models is saturated, is a consequence of superstability ([9] and [11, Theorem III.3.11]). In fact, the converse is also true [1]. Our paper provides a new first-order proof of Theorem III.3.11 of [11] when $\kappa(T) = \aleph_0$.

In abstract elementary classes (AECs), there are several approaches to generalizing superstability, and there is not yet a consensus on the correct notion. In fact it could be that superstability breaks down into several distinct dividing lines. Shelah suggests the existence of superlimits of every sufficiently large cardinality [13, Chapter N Section 2] as the definition of superstability. Elsewhere he uses frames, but in his

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categoricity transfer results (e.g. [12]) he makes use of a localized notion more similar to μ-superstability (Definition 5).

In this paper we examine the interaction, in abstract elementary classes, between μ-superstability and the statement: the union of any increasing sequence $\langle M_i \mid i < \theta\rangle$ of saturated models is saturated.

There has been much progress in understanding the interaction. We refer the reader to the introduction of [5] for an extensive review of the history of the union of saturated models and the various proposals for a definition of superstability in AECs. The most general result to date is due to Boney and Vasey for tame AECs. They prove that a version of superstability in tame abstract elementary classes implies that the union of an increasing chain of μ-saturated models is μ-saturated for $\mu > \beth_\lambda = \lambda > \mathrm{LS}(\mathcal{K})$ [5, Theorem 0.1].

We prove a related result here. Our result differs from [5] in both assumptions and methodology. We do not assume tameness, nor the existence of arbitrarily large models, and μ does not need to be large. Our methods involve limit models (and implicitly towers) and non-splitting instead of the machinery of averages and forking. Additionally our proof is shorter.

Underlying the proof of Theorem 1 are towers. A tower is a relatively new model-theoretic concept unique to abstract elementary classes. Towers were introduced by Shelah and Villaveces [14] as a tool to prove the uniqueness of limit models and later used by VanDieren [15,16] and by Grossberg, VanDieren, and Villaveces [7].

Definition 3. A *tower* is a sequence of length α of limit models, denoted by $\bar{M} = \langle M_i \in \mathcal{K}_\mu \mid i < \alpha\rangle$, along with a sequence of designated elements $\bar{a} = \langle a_i \in M_{i+1}\backslash M_i \mid i + 1 < \alpha\rangle$ and a sequence of designated submodels $\bar{N} = \langle N_i \mid i + 1 < \alpha\rangle$ for which $M_i \prec_{\mathcal{K}} M_{i+1}$, ga-tp$(a_i/M_i)$ does not μ-split over N_i, and M_i is universal over N_i (see for instance Definition I.5.1 of [15]).

Unlike many of the model-theoretic concepts in the literature of abstract elementary classes, the concept of a tower does not have a pre-established first-order analog. Therefore there is a need to understand the applications and limitations of this concept. In [18], VanDieren establishes that the statement that reduced towers are continuous is equivalent to symmetry for μ-superstable abstract elementary classes (see Fact 10). Here we further explore the connection between reduced towers and symmetry by using reduced towers in the proof of Theorem 2.

We can use Theorem 2 to weaken the assumptions of Corollary 1 of [18] by replacing categoricity in μ^+ with categoricity in μ^{+n} for some $n < \omega$ to conclude symmetry for non-μ-splitting (see Corollary 18 in Section 3). Additionally, we make progress on improving the work of [14–16,7], and [18] by proving the uniqueness of limit models of cardinality μ follows from categoricity in μ^{+n} for some $n < \omega$ without requiring tameness. The uniqueness of limit models has been explored by others, assuming tameness (e.g. [3]).

On its own, transferring symmetry is an interesting property that has been studied by others. For instance, Shelah and separately Boney and Vasey transfer symmetry in a frame between cardinals under set-theoretic assumptions [11, Section II] or using some level of tameness [5, Section 6], respectively. Our paper differs from this work in a few ways. First, we do not assume tameness nor set-theoretic assumptions, and we do not work within the full strength of a frame. The methods of this paper include reduced towers whereas the other authors use the order property as one of many mechanisms to transfer symmetry. This line of work is further extended in [22].

One of the main questions surrounding this work is the interaction between the hypothesis of μ-superstability, μ-symmetry, the uniqueness of limit models of cardinality μ, and the statement that the union of an increasing chain of μ-saturated models is μ-saturated. Theorem 1 compliments [18] where the statement, that the union of an increasing sequence $\langle M_i \in \mathcal{K}_{\mu^+} \mid i < \theta\rangle$ of saturated models is saturated, implies μ-symmetry. The following combination of Theorem 4 and Theorem 5 of [18] is close to, but not, the converse of Theorem 1.