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An intuitionistic version of Ramsey's Theorem and its use in Program Termination

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Ramsey's Theorem for pairs is a fundamental result in combinatorics which cannot be intuitionistically proved. In this paper we present a new form of Ramsey's Theorem for pairs we call the H-closure Theorem, where H stands for "homogeneous". The H-closure Theorem is a property of well-founded relations, intuitionistically provable, informative, and simple to use in intuitionistic proofs. Using our intuitionistic version of Ramsey's Theorem we intuitionistically prove the Termination Theorem by Podelski and Rybalchenko [25]. The Termination Theorem concerns an algorithm inferring termination for while-programs, and was originally proved from the classical Ramsey Theorem. Vytiniotis, Coquand, and Wahlstedt provided an intuitionistic proof of the Termination Theorem [29], using the Almost Full Theorem [11], an intuitionistic version of Ramsey's Theorem different from the H-closure Theorem. We provide a second intuitionistic proof of the Termination Theorem using the H-closure Theorem. In another paper, we use our proof to extract bounds for the Termination Theorem [5].

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1. Introduction

In computer science deciding whether a program is terminating on a given input is one of the most studied topics. In general it is a famous undecidable problem, but for some particular classes of programs it can be solved. In [25] Podelski and Rybalchenko defined a condition on well-founded relations (a generalization of a condition due to Geser [15], p. 30) and they proved that it is equivalent to the termination of transition-based programs. From this result, called the Termination Theorem, Cook, Podelski, and Rybalchenko [10] extracted an algorithm taking as input an imperative program made with the instructions **while**, **if** and assignment, and able to decide in some case whether the program is terminating or not, and in some other cases leaving the question open. The authors used in their proof of the Termination Theorem Ramsey's Theorem for pairs [26], from now on called just "Ramsey's Theorem" for short. Ramsey's Theorem is a classical result

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which cannot be intuitionistically proved: we refer to [6] for a detailed analysis of the minimal classical principle required to prove Ramsey's Theorem. According to the Π^0_2-conservativity of Classical Analysis w.r.t. Intuitionistic Analysis [14], the proof of the Termination Theorem hides some effective bounds for the while program which the theorem shows to terminate. Our long-term goal is to find them, by first turning the proof of the Termination Theorem into an intuitionistic proof. For instance, by using this proof, we can characterize the class of the primitive recursive functions in terms of Podelski and Rybalchenko's Termination Theorem [5].

Our first step is to formulate a version of Ramsey's Theorem which has a proof in intuitionistic second order logic plus basic arithmetical axiom, that is, has a proof in ordinary mathematics using only intuitionistic logic, as it is the case for proofs in Bishop's book [9]. To put otherwise, we not assume the Excluded Middle, and we do not assume other principles which are often added to Intuitionistic logic, like the Choice Axiom, Brouwer's Thesis or the Fan Theorem. Our version of Ramsey's Theorem is informative, in the sense that it has no negation, while it has a disjunction. We say that a relation R is homogeneous-well-founded, or H-well-founded for short, if the tree of all R-decreasing transitive sequences is well-founded w.r.t. the inductive definition of well-foundedness. In our version of Ramsey's Theorem, R-decreasing transitive sequences take the place of homogeneous subsets of colored graphs, and the relation R between two elements takes the place of a color for the edge connecting the two elements. We express Ramsey's Theorem as a property of well-founded relations, saying that H-well-founded relations are closed under finite unions. For short we call this statement *H-closure*. Thus, we are able to split the proof of Ramsey's Theorem into two parts: the intuitionistic proof of H-closure, followed by some "simple" (in the sense of the Reverse Mathematics, see [8]) classical proof of the equivalence between Ramsey's Theorem and H-closure.

The result closest to the H-closure Theorem we could find is the Almost Full Theorem by Coquand [11]. Coquand, as Veldman and Bezem did before him [28], considers *almost full relations* (a kind of dual of H-closed relations) and proves that they are closed under finite *intersections*. Veldman and Bezem use the Choice Axiom of type 0 (if $\forall x \in \mathbb{N}.\exists y \in \mathbb{N}.C(x, y)$, then $\exists f : \mathbb{N} \to \mathbb{N}.\forall x \in \mathbb{N}.C(x, f(x))$) and Brouwer's thesis. Coquand's proof, instead, is purely intuitionistic, and it may be used to give a purely intuitionistic proof of the Termination Theorem [29]. However, it is not evident what are the effective bounds hidden in Coquand's proof of the Termination Theorem. If we compare the H-closure Theorem with the Almost Full Theorem, in the most recent version by Coquand [11], we find no easy way to intuitionistically deduce one from the other, due to the use of de' Morgan's Law to move from the definition of almost full to the definition of H-closure. H-closure is in a sense more similar to the original Ramsey's Theorem, because it was obtained from it with just one classical step, a contrapositive (see Section 2), while almost fullness requires one application of de' Morgan's Law, followed by a contrapositive. We expect that H-closure, hiding one application less of de' Morgan's Law, should be a version of Ramsey's Theorem simpler to use in intuitionistic proofs and for extracting bounds, as we did in [5].

Another motivation for our work is the following. In [19] Lee, Jones and Ben-Amram introduced the notion of size-change termination and they proved the Size-Change Termination Theorem, SCT Theorem for short, which states that a first order functional program is terminating if and only if it satisfies a property, called SCT, which can be statically verified from the recursive definition of the program. Also in this proof the authors used Ramsey's Theorem for pairs. The authors of [29] provided an intuitionistic proof of the SCT Theorem. The second author of this paper found a very different proof of the SCT Theorem which uses H-closure [27].

Our paper is an expanded version of the conference paper [7]. This is the plan of the paper. In Section 2 we present Ramsey's Theorem for pairs and we informally introduce the H-closure Theorem. In Section 3 we formally define inductive well-foundedness and H-well-foundedness, whose main properties are stated in Section 4. The goal of Section 5 is to present what we call *Intuitionistic Nested Fan Theorem*, which is a part of the proof of the H-closure Theorem, as shown in Section 6. In Section 7 we intuitionistically prove the Termination Theorem, using intuitionistic generalized inductive definitions (a fragment of second order

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