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The structure of the Mitchell order – II

Omer Ben-Neria [1]

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ABSTRACT

We address the question regarding the structure of the Mitchell order on normal measures. We show that every well founded order can be realized as the Mitchell order on a measurable cardinal κ, from some large cardinal assumption.

1. Introduction

In this paper we address the question regarding the possible structure of the Mitchell order $\triangleleft$, at a measurable cardinal κ. The Mitchell order was introduced by William Mitchell in [13], who showed it is a well-founded order. The question whether every well-founded order can be realized as the Mitchell order on the set of some specific measurable cardinal, has been open since. By combining various forcing techniques with inner model theory we succeed in constructing models which realize every well-founded order as the Mitchell order on a measurable cardinal.

Given two normal measures U, W, we write $U \triangleleft W$ to denote that $U \in M_W \cong \mathrm{Ult}(V, W)$. For every measurable cardinal κ, let $\triangleleft(\kappa)$ be the restriction of $\triangleleft$ to the set of normal measures on κ, and let $o(\kappa) = \mathrm{rank}(\triangleleft(\kappa))$ be its (well-foundedness) rank.

The research on the possible structure on the Mitchell order $\triangleleft(\kappa)$ is closely related to the question of its possible size, namely, the number of normal measures on κ: The first results by Kunen [8] and by Kunen and Paris [9] showed that this number can take the extremal values of 1 and κ^{++} (in a model of GCH) respectively. Soon after, Mitchell [13,14] showed that this size can be any cardinal λ between 1 and κ^{++}, under the large cardinal assumption and in a model of $o(\kappa) = \lambda$. Baldwin [2] showed that for $\lambda < \kappa$ and from

E-mail address: omerbe3@tau.ac.il.
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stronger large cardinal assumptions, κ can also be the first measurable cardinal. Apter–Cummings–Hamkins [1] proved that there can be κ^+ normal measures on κ from the minimal assumption of a single measurable cardinal; for $\lambda < \kappa^+$, Leaning [10] reduced the large cardinal assumption from $o(\kappa) = \lambda$ to an assumption weaker than $o(\kappa) = 2$. The question of the possible number of normal measures on κ was finally resolved by Friedman and Magidor in [7], where it is shown that κ can carry any number of normal measures $1 \leq \lambda \leq \kappa^{++}$ from the minimal assumption. The Friedman–Magidor method will be extensively used in this work.

Further results were obtained on the possible structure of the Mitchell order: Mitchell [13] and Baldwin [2] showed that from some large cardinal assumptions, every well-order and pre-well-order (respectively) can be isomorphic to $\triangleleft(\kappa)$ at some κ. Cummings [5,6], and Witzany [19] studied the $\triangleleft$ ordering in various generic extensions, and showed that $\triangleleft(\kappa)$ can have a rich structure. Cummings constructed models where $\triangleleft(\kappa)$ embeds every order from a specific family of orders we call tame. Witzany showed that in a Kunen–Paris extension of a Mitchell model $L[\mathcal{U}]$, with $o^{\mathcal{U}}(\kappa) = \kappa^{++}$, every well-founded order of cardinality $\leq\kappa^+$ embeds into $\triangleleft(\kappa)$.

In this paper, the main idea for realizing well-founded orders as $\triangleleft(\kappa)$ is to force over an extender model $V = L[E]$ with a sufficiently rich $\triangleleft$ structure on a set of extenders at κ. By forcing over V we can collapse the generators of these extenders, giving rise to extensions of these extenders, which are equivalent to ultrafilters on κ. The possible structure of the Mitchell order on arbitrary extenders was previously studied by Steel [18] and Neeman [15] who showed that the well-foundedness of the Mitchell order fails exactly at the level of a rank-to-rank extender.

For the most part, the extenders on κ which will be used, do not belong to the main sequence E. Rather, they are of the form $F' = i_\theta(F)$, where $F \in E$ overlaps a measure on a cardinal $\theta > \kappa$, and i_θ is an elementary embedding with $\text{cp}(i_\theta) = \theta$. There is a problem though; the extenders F' may not be κ-complete. To solve this we incorporate an additional forcing extension by which F' will generically regain its missing sequence of generators.[2] Forcing the above would translate the $\triangleleft$ on certain extenders F', to $\triangleleft(\kappa)$; however some of the normal measures in the resulting model will be unnecessary and will need to be destroyed. This will be possible since the new normal measures on κ are separated by sets[3] allowing us to remove the undesired normal measures in an additional generic extension, which we refer to as a *final cut*.

The combination of these methods will be used to prove the main result:

Theorem 1.1. *Let $V = L[E]$ be a core model. Suppose that κ is a cardinal in V and $(S, <_S)$ is a well-founded order of cardinality $\leq \kappa$, so that*

1. *There are at least $|S|$ measurable cardinals above κ; let θ be the supremum of the successors of the first $|S|$,*
2. *There is a $\triangleleft$-increasing sequence of $(\theta + 1)$-strong extenders $\vec{F} = \langle F_\alpha \mid \alpha < \text{rank}(S, <_S)\rangle$*

Then there is a generic extension V^* of V in which $\triangleleft(\kappa)^{V^*} \cong (S, <_S)$.

In particular, if E contains a $\triangleleft$-increasing sequence of extenders $\vec{F} = \langle F_\alpha \mid \alpha < \kappa^+\rangle$, so that each F_α overlaps the first κ measurable cardinals above κ, then every well founded order $(S, <_S) \in V$ of cardinality $\leq\kappa$ can be realized as $\triangleleft(\kappa)$ in a generic extension of V. As an immediate corollary of the proof we have that under slightly stronger large cardinal assumptions, including a class of cardinals κ carrying similar overlapping extenders, there is a class forcing extension V^* in which every well founded order $(S, <_S)$ is isomorphic to $\triangleleft(\kappa)$ for some κ.

This paper is the second of a two-parts study on $\triangleleft$. In the first part [4], a wide family of well-founded orders named tame orders, was isolated and it was shown that every tame order of cardinality at most κ can be realized from an assumption weaker than $o(\kappa) = \kappa^+$.

[2] I.e., while $M_{F'} \cong \text{Ult}(V, F')$ may not be closed under κ-sequences, its embedding $j_{F'} : V \to M_{F'}$ will extend in $V[G]$ to $j' : V[G] \to M_{F'}[G']$ so that ${}^{\kappa}M_{F'}[G'] \subset M_{F'}[G']$.

[3] Namely, we can associate to each normal measure U on κ a set $X_U \subset \kappa$, which is not contained in any distinct normal measure.