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The automorphism group of the enumeration degrees ☆

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ABSTRACT

We investigate the extent to which Slaman and Woodin's framework for the analysis of the automorphism group of the structure of the Turing degrees can be transferred to analyze the automorphism group of the structure of the enumeration degrees.

1. Introduction

A goal of computability theory is to give a mathematical account of a structure, which arises as a formal way of classifying the relative computational strength of objects. The most studied example of such structures is that of the Turing degrees, $\mathcal{D}_T$, arising from the notion of Turing reducibility. To understand such a structure we study its complexity: how rich is it algebraically; how complicated is its theory; what relations are first order definable in it; does it have nontrivial automorphisms. In the study of $\mathcal{D}_T$ we find that all these questions are interrelated in a very strong way. The definability of the jump operator by Slaman and Shore [15] relies on a method used by Slaman and Woodin [17] to analyze the automorphism group of $\mathcal{D}_T$, $Aut(\mathcal{D}_T)$. This analysis reveals a strong connection between the definability properties of the structure of the Turing degrees and second order arithmetic, leading to Slaman and Woodin's famous *Biinterpretability conjecture*. Even though this analysis does not give a complete answer to the question of the existence of nontrivial automorphisms of $\mathcal{D}_T$, it sheds light on the properties of $Aut(\mathcal{D}_T)$: it is shown

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that the group is at most countable, in fact that all its members are arithmetically definable and every automorphism is completely determined by its action on a single element.

A different approach to the analysis of the structure of the Turing degrees is to study it within a richer context, a context which would hopefully reveal new hidden relationships. Such a context is the structure of the enumeration degrees $\mathcal{D}_e$. This structure is as well an upper semi-lattice with jump operation, induced by a weaker form of relative computability: a set A is enumeration reducible to a set B if every enumeration of the set B can be effectively transformed into an enumeration of the set A. The Turing degrees as an upper semi-lattice with jump operation can be embedded in the enumeration degrees, and thus can be studied as a substructure of the enumeration degrees, the substructure $\mathcal{TOT}$ of the total enumeration degrees.

Working in $\mathcal{D}_e$ is different. The enumeration degrees are not closed under complement. This makes coding and relativization significantly more complicated. On the other hand in $\mathcal{D}_e$ we find that certain definability results have more natural proofs. Kalimullin [8] discovered the existence of pairs of enumeration degrees, called $\mathcal{K}$-pairs, with structural properties reminiscent of structural properties of generic Turing degrees, and showed that the class of such pairs of enumeration degrees is first order definable in $\mathcal{D}_e$, by a simple Π_1 statement in the language of lattices. The existence of $\mathcal{K}$-pairs is unique to the structure of the enumeration degrees. In fact their existence in the structure $\mathcal{D}_T$ would contradict the precise property of the Turing degrees that allows Slaman and Shore to extract the definition of the Turing jump from the automorphism analysis. This shows that the method devised by Slaman and Woodin to investigate $Aut(\mathcal{D}_T)$ cannot be transferred directly to analyze $Aut(\mathcal{D}_e)$. Still, the first step has been made. In [16] Slaman and Woodin prove the Coding Theorem for the structure $\mathcal{D}_e$, and as a consequence show that the first order theory of $\mathcal{D}_e$ is computably isomorphic to that of second order arithmetic.

Kalimullin [8] showed that the first order definability of the enumeration jump follows nevertheless from the existence of $\mathcal{K}$-pairs. A consequence to this is the first order definability of the total enumeration degrees above $\mathbf{0}_e'$. Ganchev and Soskova [3] bring down the definition of $\mathcal{K}$-pairs to a local level, showing that the notion is still definable in the local structure of the enumeration degrees bounded by $\mathbf{0}_e'$. In [5] they combine $\mathcal{K}$-pairs and the Coding Theorem to show that the first order theory of the local structure is computably isomorphic to first order arithmetic. In [6] Ganchev and Soskova investigate maximal $\mathcal{K}$-pairs and show that within the local structure the total degrees are first order definable. The question of the global definability of $\mathcal{TOT}$, set first by Rogers [12], remains unanswered and seems to play a central role in this puzzle. Rozinas [13] showed that $\mathcal{TOT}$ is an automorphism base for $\mathcal{D}_e$. Thus if $\mathcal{TOT}$ were definable then a negative answer to the question of the rigidity of $\mathcal{D}_e$ would yield a negative answer to the same question for $\mathcal{D}_T$.

We will outline how to adapt Slaman and Woodin's framework to investigate the properties of $Aut(\mathcal{D}_e)$. We shall show that $Aut(\mathcal{D}_e)$ is at most countable, that its members are arithmetically definable and that $\mathcal{D}_e$ has an automorphism base consisting of a single element. This analysis in $\mathcal{D}_e$ brings us one step closer to the definability of $\mathcal{TOT}$, namely one parameter away from it.

2. Preliminaries

The notions and definitions that will be used in this article come from various parts of logic and we will not be able to give definitions and outline their basic properties. We hope to be able to give sufficient references, to books and articles, where these notions are explained in detail. A main reference for this article is the Slaman and Woodin's manuscript [17]. We will give definitions to all notions that are not defined in [17] and are used below.

Definition 1. A set A is *enumeration reducible* ($\leq_e$) to a set B if there is a c.e. set Γ such that:

$$A = \Gamma(B) = \{n \mid \exists u(\langle n, u\rangle \in \Gamma \ \& \ D_u \subseteq B)\},$$