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Model theory of special subvarieties and Schanuel-type conjectures

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A B S T R A C T

We use the language and tools available in model theory to redefine and clarify the rather involved notion of a special subvariety known from the theory of Shimura varieties (mixed and pure).

1. Introduction

1.1. The first part of the paper (section 2) is essentially a survey of developments around the program outlined in the talk to the European Logic Colloquium-2000 and publication [34]. It then continues with new research which aims, on the one hand side, to extend the model-theoretic picture of [34] and of section 2 to the very broad mathematical context of what we call here *special coverings of algebraic varieties*, and on the other hand, to use the language and the tools available in model theory to redefine and clarify the rather involved notion of a *special subvariety* known from the theory of Shimura varieties (mixed and pure) and some extensions of this theory.

Our definition of special coverings of algebraic varieties includes semi-abelian varieties, Shimura varieties (definitely the pure ones, and we also hope but do not know if the mixed ones in general satisfy all the assumptions) and much more, for example, the Lie algebra covering a simple complex of Lie group $\mathrm{SL}(2,\mathbb{C})$.

Recall from the discussion in [34] that our specific interest in these matters arose from the connection to Hrushovski's construction of *new* stable structures (see e.g. [18]) and their relationship with generalised Schanuel conjectures. This subject is also closely related to the Trichotomy Principle and Zariski geometries.

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In the current paper we establish that the geometry of an arbitrary special covering of an algebraic variety is controlled by a Zariski geometry the closed subsets of which we call (weakly) special. The combinatorial type of simple (i.e. strongly minimal) weakly special subsets are classifiable by the Trichotomy Principle. Using this geometry and related dimension notions we can define a corresponding very general analogue of "Hrushovski's predimension" and formulate corresponding "generalised Schanuel's conjecture" as well as a very general forms of André–Oort, the CIT and Pink's conjectures (Zilber–Pink conjectures). Note that in this generality one can see a considerable overlap of the generalised Schanuel conjectures with the André conjecture on periods [1] (generalising the Grothendieck period conjecture) which prompt further questions on the model-theoretic nature of fundamental mathematics.

2. Analytic and pseudo-analytic structures

Recall that a strongly minimal structure $\mathbf{M}$ (or its theory) can be given a coarse classification by the type of *the combinatorial geometry* that is induced by the pregeometry (M, acl) on the set $[M \setminus \mathrm{acl}(\emptyset)]/_{\sim}$, where $x \sim y$ iff $\mathrm{acl}(x) = \mathrm{acl}(y)$.

The Trichotomy conjecture by the author stated that for any strongly minimal structure $\mathbf{M}$ the geometry of $\mathbf{M}$ is either trivial or linear (the two united under the name *locally modular*), or the geometry of $\mathbf{M}$ is the same as of an algebraically closed field, and in this case the structure $\mathbf{M}$ is bi-interpretable with the structure of the field.

E. Hrushovski refuted this conjecture in the general setting [18]. Nevertheless the conjecture was confirmed, by Hrushovski and the author in [19], for an important subclass of structures, Zariski geometries, except for the clause stating the bi-interpretability, where the situation turned out to be more delicate.

2.1. Recall that the main suggestion of [34] was to treat an (amended version of) Hrushovski's counter-examples as *pseudo-analytic* structures, analogues of classical analytic structures. Hrushovski's predimension, and the corresponding inequality $\delta(X) \geq 0$, a key ingredient in the construction, can be seen then to directly correspond to certain type of conjectures of transcendental number theory, which we called *Generalised Schanuel* conjectures.

The ultimate goal in classifying the above mentioned pseudo-analytic structures has been to give a (non-first-order) axiomatisation and prove a categoricity theorem for the axiomatisable class.

2.2. The algebraically closed fields with pseudo-exponentiation, $\mathrm{F}_{\exp} = (\mathrm{F};, +, \cdot, \exp)$, analogues of the classical structure $\mathbb{C}_{\exp} = (\mathbb{C};, +, \cdot, \exp)$, was the first class studied in detail.

The axioms for $\mathrm{F}_{\exp}$ are as follows.

ACF_0: F is an algebraically closed fields of characteristic 0;

EXP: $\exp : \mathbb{G}_a(\mathrm{F}) \to \mathbb{G}_m(\mathrm{F})$
is a surjective homomorphism from the additive group $\mathbb{G}_a(\mathrm{F})$ to the multiplicative group $\mathbb{G}_m(\mathrm{F})$ of the field F, and

$$\ker \exp = \omega\mathbb{Z}, \text{ for some } \omega \in \mathrm{F};$$

SCH: for any finite X,

$$\delta(X) := \mathrm{tr.deg}_{\mathbb{Q}}(X \cup \exp(X)) - \mathrm{ldim}_{\mathbb{Q}}(X) \geq 0.$$

Here $\mathrm{tr.deg}(X)$ and $\mathrm{ldim}_{\mathbb{Q}}(X)$ are the transcendence degree and the dimension of the $\mathbb{Q}$-linear space spanned by X, dimensions of the classical pregeometries associated with the field F, and $\delta(X)$ takes the role of *Hrushovski's predimension*, which gives rise to a new dimension notion and new pregeometry following