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On Δ^0_2-categoricity of equivalence relations ☆

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ABSTRACT

We investigate which computable equivalence structures are isomorphic relative to the Halting problem.

1. Introduction

This paper is within the scope of two frameworks: the first one investigates effective properties of equivalence relations (to be discussed in Section 1.1), and the other one studies non-computable isomorphisms between computable structures (see Section 1.2). The main idea of this paper can be described as follows: We view a computable equivalence structure as an abstraction to the situation when a computable algebraic structure has several *components*. Examples of such structures include direct or cardinal sums of groups or rings, shuffle and free sums of Boolean algebras, and graphs having several connected components. We simplify the situation by essentially removing all algebraic content from each component, so that we have to care only about matching the sizes of components correctly when we construct an isomorphism. The idea is that to understand the general situation, we should *first* understand the much simpler associated setting where the algebra has been stripped away. In particular, the present paper is a companion to Downey, Mel-

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nikov and Ng [15], where p-groups are associated with equivalence relations. One might expect that it would be easy to understand Δ^0_2-isomorphisms for these "degenerate" computable structures, particularly ones as simple as equivalence relations. We will see that the subject is a lot deeper than one might expect. Having abstracted the algebraic properties into a setting with no apparent *algebraic* difficulties, now one faces many *computability-theoretic* difficulties in such studies. Indeed, we see that a non-standard $\emptyset'''$-technique is required to answer a very basic (but fundamental) question. The proof is of some purely technical interest; its high complexity also partially explains why so little is known about Δ^0_2-isomorphisms between computable structures in general. We now turn to a more detailed discussion and background.

1.1. Effectively presentable equivalence structures

Arguably, the study of effective reducibilities between countable equivalence relations goes back to Mal'cev who founded the theory of numberings (see Ershov [16] for a detailed exposition). Many results of numbering theory can be translated into results on equivalence relations and vice versa, see the recent paper [1] for more details. Numbering theory has been one of the central topics in the Soviet logic school for over 40 years. In the West, the topic has traditionally received less attention (but see Lachlan [25]), and it is fair to say that it did not occupy center stage in computability theory.

Recently however, the subject has enjoyed a rapid development, partially because of the simultaneous and successful development of the theory of Borel equivalence relations, see textbook [6]. The theory of effective equivalence relations has grown to a rather broad area; we cite [1,18,9] for recent results on this subject. Many results of this paper can be stated in terms of Δ^0_2-embeddings between effectively presented equivalence structures. However, we choose a different approach (see the next subsection) and thus we will not provide any further background on effective reducibilities between equivalence structures.

1.2. Non-computable isomorphisms between computable structures

Recall that a structure is *computable* if its open diagram is a computable set. Recall that a computable algebraic structure $\mathcal{A}$ is *computably categorical* if any computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$ is *computably* isomorphic to $\mathcal{A}$. In many common classes computable categoricity can be understood as a synonym of being algebraically tame. For example, it is well-known that a computable linear order is computably categorical iff it has finitely many adjacencies, a computable Boolean algebra is computably categorical off it has finitely many atoms, and there is a full and simple description of computably categorical abelian p-groups, see [27,31,32] and [3,17] for further examples. Most of these classes are not effectively universal (i.e., these structures cannot effectively encode an arbitrary computable arbitrary graph, see [11]). On the other hand, when we want to study more complex algebraic structures and their computability theory, we often need to abandon computable categoricity. As a consequence, there has been an increasing interest in *non-computable* isomorphisms between computable structures.

The central notion in the study of non-computable isomorphisms is:

Definition 1.1. Let $n > 1$ be a natural number. A computable algebraic structure A is Δ^0_n-categorical if every two computable presentations of A are $0^{(n-1)}$-isomorphic.[1]

In contrast to computable categoricity, obtaining a complete classification of Δ^0_n-categoricity in a given class is typically a difficult task. Already for $n = 2$ and even for algebraically very well understood classes,

[1] Here $0^{(n+1)}$ stands for the $(n+1)$th iterate of the Halting problem. We note that there are variations of Definition 1.1 such as the notion of relative Δ^0_n-categoricity [3], and also related notions of categoricity spectra [20] and degrees of categoricity [19,10].