Annals of Pure and Applied Logic 166 (2015) 881–933

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Rules with parameters in modal logic I

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ARTICLE INFO

Article history:
Received 17 June 2013
Received in revised form 1 April 2015
Accepted 2 April 2015
Available online 8 May 2015

MSC:
primary 03B45, 12X34
secondary 08B20, 68T15

Keywords:
Modal logic
Admissible rule
Equational unification

ABSTRACT

We study admissibility of inference rules and unification with parameters in transitive modal logics (extensions of **K4**), in particular we generalize various results on parameter-free admissibility and unification to the setting with parameters.
Specifically, we give a characterization of projective formulas generalizing Ghilardi's characterization in the parameter-free case, leading to new proofs of Rybakov's results that admissibility with parameters is decidable and unification is finitary for logics satisfying suitable frame extension properties (called cluster-extensible logics in this paper). We construct explicit bases of admissible rules with parameters for cluster-extensible logics, and give their semantic description. We show that in the case of finitely many parameters, these logics have independent bases of admissible rules, and determine which logics have finite bases.
As a sideline, we show that cluster-extensible logics have various nice properties: in particular, they are finitely axiomatizable, and have an exponential-size model property. We also give a rather general characterization of logics with directed (filtering) unification.
In the sequel, we will use the same machinery to investigate the computational complexity of admissibility and unification with parameters in cluster-extensible logics, and we will adapt the results to logics with unique top cluster (e.g., **S4.2**) and superintuitionistic logics.

1. Introduction

Admissibility of inference rules is among the fundamental properties of nonclassical propositional logic: a rule is admissible if the set of tautologies of the logic is closed under the rule, or equivalently, if adjunction of the rule to the logic does not lead to derivation of new tautologies. Admissible rules of basic transitive modal logics ($\mathbf{K4}, \mathbf{S4}, \mathbf{GL}, \mathbf{Grz}, \mathbf{S4.3}, \dots$) are fairly well understood. Rybakov proved that admissibility in a large class of modal logics is decidable and provided semantic description of their admissible rules, see [27] for a detailed treatment. Ghilardi [8] gave a characterization of projective formulas in terms of extension

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http://dx.doi.org/10.1016/j.apal.2015.04.004

properties of their models, and proved the existence of finite projective approximations. This led to an alternative proof of some of Rybakov's results, and it was utilized by Jeřábek [15,18] to construct explicit bases of admissible rules, and to determine the computational complexity of admissibility [17]. A sequent calculus for admissible rules was developed by Iemhoff and Metcalfe [13]. Methods used for transitive modal logics were paralleled by a similar treatment of intuitionistic and intermediate logics, see e.g. [27,7,11,12].

Admissibility is closely related to unification [2,1]: for equational theories corresponding to algebraizable propositional logics, E-unification can be stated purely in terms of logic, namely a unifier of a formula is a substitution which makes it a tautology. Thus, a rule is admissible iff every unifier of the premises of the rule also unifies its conclusion, and conversely the unifiability of a formula can be expressed as nonadmissibility of a rule with inconsistent conclusion. In fact, the primary purpose of Ghilardi [8] was to prove that unification in the modal logics in question is finitary.

In unification theory, it is customary to work in a more general setting that allows for extension of the basic equational theory by free constants. In logical terms, formulas may include atoms (variously called parameters, constants, coefficients, or metavariables) that behave like ordinary propositional variables for most purposes, but are required to be left fixed by substitutions. Some of the above-mentioned results on admissibility in transitive modal logics also apply to admissibility and unification with parameters, in particular Rybakov [24–27] proved the decidability of admissibility with parameters in basic transitive logics, and he has recently extended his method to show that unification with parameters is finitary in these logics [28,29]. Nevertheless, a significant part of the theory only deals with parameter-free rules and unifiers.

The purpose of this paper is to (at least partially) remedy this situation by extending some of the results on admissibility in transitive modal logics to the setup with parameters. Our basic methodology is similar to the parameter-free case, however the presence of parameters brings in new phenomena leading to nontrivial technical difficulties that we have to deal with.

For a more detailed overview of the content of the paper, after reviewing basic concepts and notation (Section 2) and establishing some elementary background on multiple-conclusion consequence relations with parameters (Section 2.1), we start in Section 3 with a parametric version of Ghilardi's characterization of projective formulas in transitive modal logics with the finite model property in terms of a suitable model extension property on finite models. In Section 4, we introduce the class of cluster-extensible (clx) logics (and more generally, Par-extensible logics for the case when the set Par of allowed parameters is finite), and we use the results from Section 3 to show that in clx (or Par-extensible) logics, all formulas have projective approximations. As a corollary, this reproves results of Rybakov [27,28] that such logics L have finitary unification type, and if L is decidable, then admissibility in L is also decidable, and one can compute a finite complete set of unifiers of a given formula. In order to determine which of these logics have unitary unification, we include in Section 4.2 a simple syntactic criterion for directed (filtering) unification, vastly generalizing the result of Ghilardi and Sacchetti [9]. In Section 4.3, we look more closely at semantic and structural properties of clx logics: we show that every clx logic is finitely axiomatizable, decidable, $\forall\exists$-definable on finite frames, and has an exponential-size model property. Moreover, the class of clx logics is closed under joins in the lattice of normal extensions of **K4**. (These results mostly do not have good analogues in the parameter-free case; they exploit the fact that the extension conditions designed to make the other results on admissibility and unification work need to be more restrictive when parameters are considered.)

In Section 5, we introduce (multiple-conclusion) rules corresponding to the existence of a parametric version of tight predecessors, generalizing the parameter-free rules considered in [15,18]. We investigate their semantic properties, and as the main result of this section, we show that these extension rules form bases of admissible rules for clx or Par-extensible logics. We present single-conclusion variants of these bases in Section 5.1. Finally, in Section 5.2, we modify the extension rules further to provide independent bases of admissible rules with finitely many parameters for Par-extensible logics, and we show that finite bases exist if and only if the logic has bounded branching.