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Easton's theorem for Ramsey and strongly Ramsey cardinals

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ABSTRACT

We show that, assuming GCH, if κ is a Ramsey or a strongly Ramsey cardinal and F is a class function on the regular cardinals having a closure point at κ and obeying the constraints of Easton's theorem, namely, $F(\alpha) \leq F(\beta)$ for $\alpha \leq \beta$ and $\alpha < \mathrm{cf}(F(\alpha))$, then there is a cofinality-preserving forcing extension in which κ remains Ramsey or strongly Ramsey respectively and $2^\delta = F(\delta)$ for every regular cardinal δ.

1. Introduction

Since the earliest days of set theory, when Cantor put forth the Continuum Hypothesis in 1877, set theorists have been trying to understand the properties of the continuum function dictating the sizes of powersets of cardinals. In 1904, König presented his false proof that the continuum is not an aleph, from which Zermelo derived the primary constraint on the continuum function, the Zermelo–König inequality, that $\alpha < \mathrm{cf}(2^\alpha)$ for any cardinal α. In the following years, Jourdain and Hausdorff introduced the Generalized Continuum Hypothesis, and in another two decades Gödel showed the consistency of GCH by demonstrating that it held in his constructible universe L.[1] The full resolution to the question of CH in ZFC had to wait for Cohen's development of forcing in 1963, which could be used to construct set-theoretic universes

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[1] For a full account of the early history of the GCH see [21].

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with arbitrarily large sizes of the continuum. Gödel's and Cohen's results together finally established the independence of CH from ZFC. A decade later, building on advances in forcing techniques, Easton showed that, assuming GCH, any class function F on the regular cardinals satisfying $F(\alpha) \leq F(\beta)$ for $\alpha \leq \beta$ and $\alpha < \mathrm{cf}(F(\alpha))$ can be realized as the continuum function in a cofinality-preserving forcing extension [5], so that in the extension $2^\delta = F(\delta)$ for all regular cardinals δ. Thus, any desired monotonous function on the regular cardinals satisfying the necessary constraints of the Zermelo–König inequality could be realized as the continuum function in some set-theoretic universe.[2]

For some simple and other more subtle reasons, the presence of large cardinals in a set-theoretic universe imposes additional constraints on the continuum function, the most obvious of these being that the continuum function must have a closure point at any inaccessible cardinal. Other restrictions arise from large cardinals with strong reflecting properties. For instance, GCH cannot fail for the first time at a measurable cardinal, although Levinski showed in [18] that GCH can hold for the first time at a measurable cardinal. Supercompact cardinals impose much stronger constraints on the continuum function. If κ is supercompact and GCH holds below κ, then it must hold everywhere and, in contrast to Levinski's result, if GCH fails for all regular cardinals below κ, then it must fail for some regular cardinal $\geq \kappa$ [15].[3] Additionally, certain continuum patterns at a large cardinal can carry increased consistency strength as, for instance, a measurable cardinal κ at which GCH fails has the consistency strength of a measurable cardinal of Mitchell order $o(\kappa) = \kappa^{++}$ [9]. Some global results are also known concerning sufficient restrictions on the continuum function in universes with large cardinals. Menas showed in [19] that, assuming GCH, there is a cofinality-preserving and supercompact cardinal preserving forcing extension realizing any *locally definable*[4] function on the regular cardinals obeying the constraints of Easton's theorem, and Friedman and Honzik extended this result to strong cardinals using generalized Sacks forcing [7]. In [2], Cody showed that if GCH holds, and if F is any function obeying the constraints of Easton's theorem (F need not be locally definable) such that each Woodin cardinal is closed under F, then there is a cofinality-preserving forcing extension realizing F to which all Woodin cardinals are preserved.

Definition 1.1. We say that a (possibly class) function F is a *possible continuum function* if its domain is contained in the class of regular cardinals and for all $\alpha \leq \beta$ in the domain of F, we have $F(\alpha) \leq F(\beta)$ and $\alpha < \mathrm{cf}(F(\alpha))$.

In this article, we show that, assuming GCH, if κ is a Ramsey or a strongly Ramsey cardinal, then any possible continuum function with domain regular cardinals $\leq\kappa$ and a closure point at κ is realized as the continuum function in a cofinality-preserving forcing extension in which κ remains Ramsey or strongly Ramsey respectively. In particular, this extends Levinski's result mentioned earlier to Ramsey and strongly Ramsey cardinals. Strongly Ramsey cardinals, introduced by Gitman in [11], fall in between Ramsey cardinals and measurable cardinals in consistency strength, and we shall review their properties in Section 2.

Main Theorem. *Assuming* GCH, *if κ is a Ramsey or a strongly Ramsey cardinal and F is a possible continuum function defined on the regular cardinals $\leq\kappa$ having a closure point at κ, then there is a cofinality-preserving forcing extension in which κ remains Ramsey or strongly Ramsey respectively, and F is realized as the continuum function on the regular cardinals $\delta \leq \kappa$, namely $2^\delta = F(\delta)$.*

[2] The situation with singular cardinals turned out to be much more complicated. Silver showed, for example, that if δ is a singular cardinal of an uncountable cofinality and $2^\alpha = \alpha^+$ for all $\alpha < \delta$, then $2^\delta = \delta^+$ and thus, there is not the same extent of freedom for the continuum function on singular cardinals [22].

[3] Interestingly, in the absence of the axiom of choice, the existence of measurable or supercompact cardinals does not impose any of these restrictions on the continuum function [1].

[4] A function F is *locally definable* if there is a true sentence ψ and a formula $\varphi(x, y)$ such that for all cardinals γ, if $H_\gamma \models \psi$, then F has a closure point at γ and for all $\alpha, \beta < \gamma$, we have $F(\alpha) = \beta \leftrightarrow H_\gamma \models \varphi(\alpha, \beta)$.