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Fragility and indestructibility II

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ABSTRACT

In this paper we continue work from a previous paper on the fragility and indestructibility of the tree property. We present the following:

(1) A preservation lemma implicit in Mitchell's PhD thesis, which generalizes all previous versions of Hamkins' Key lemma.
(2) A new proof of the 'superdestructibility' theorems of Hamkins and Shelah.
(3) An answer to a question from our previous paper on the apparent consistency strength of the assertion "The tree property at $\aleph_2$ is indestructible under $\aleph_2$-directed closed forcing".
(4) Two models for successive failures of weak square on long intervals of cardinals.

Techniques for preserving the tree property are central to a growing literature of consistency results obtaining the tree property as successive regular cardinals (see for example [11,1,3,15,17]). These techniques can be viewed abstractly as indestructibility results, which typically arise from either integration of preparation forcing or preservation lemmas. In our original paper [19] we proved results using both of these methods. In particular, using methods of Abraham and Cummings and Foreman [3,1] we showed that modulo the existence of a supercompact cardinal it is consistent that the tree property holds at ω_2 and is indestructible under ω_2-directed closed forcing. Further, by proving a new preservation lemma we showed that the tree property at ω_2 in a model of Mitchell [11] is indestructible under the forcing to add an arbitrary number of Cohen reals. It follows that the tree property at ω_2 is consistent with $2^\omega > \omega_2$. The preservation lemma was

Lemma 0.1. *Let τ, η be cardinals with η regular and $2^\tau \geq \eta$. Let $\mathbb{P}$ be τ^+-cc and $\mathbb{R}$ be τ^+-closed. Let $\dot{T}$ be a $\mathbb{P}$-name for an η-tree. Then in $V[\mathbb{P}]$ forcing with $\mathbb{R}$ cannot add a branch through T.*

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This lemma was applied in work of Neeman [15] and inspired a lemma of Sinapova [16]. Indeed the idea that for some preservation properties the forcing only needs to be *formerly* closed seems to be quite powerful. This paper provides further applications of the above lemma, this time to obtain successive failures of weak square.

The subject of fragility is more subtle. In the original paper we proved that in Mitchell's model there is a cardinal preserving forcing which adds $\square_{\omega_1}$. In some sense this is the easiest kind of fragility result. The forcing is specifically designed to add a strong sufficient condition for the existence of an Aronszajn tree. More interesting fragility results consist in proving that the tree property fails in an extension where we hoped that it might hold. In this paper we give a definition of an Aronszajn tree which occurs as the tree of attempts to construct an object in some generic extension. This gives a new proof of theorems of Hamkins, and Hamkins and Shelah and an instance of fragility where we do not force a strong sufficient condition.

The results of the current paper are as follows:

- In Section 1 we prove a strong generalization of Hamkins' Key lemma and provide a few applications. We also remark on a few applications of a related lemma appearing in another of the author's papers.
- In Section 2 we provide new proofs of the main theorems of Hamkins [5], and Hamkins and Shelah [7]. As a special case of Theorem 2.8, we have that if κ is inaccessible, $\mathbb{P}$ has size less than κ and it is forced by $\mathbb{P}$ that $\dot{\mathbb{Q}}$ is κ-closed and adds a subset to κ, then in $V[\mathbb{P} * \dot{\mathbb{Q}}]$ there is a κ-Aronszajn tree. The more general statement gives the result of Hamkins and Shelah.
- In Section 3 we answer a question from our previous fragility and indestructibility paper on the apparent consistency strength of the statement "The tree property at ω_2 is indestructible under ω_2-directed closed forcing". In particular we show that any reasonable forcing to obtain the consistency of this statement requires a strongly compact cardinal.
- In Section 4 we prove some results which are generalizations of a result claimed without proof by Mitchell [11]. In particular we show that if there are infinitely many Mahlo cardinals, then there is a forcing extension in which $\square^*_{\aleph_n}$ fails for $1 \leq n < \omega$. In this model we achieve the most economical failure of GCH possible. In particular $2^{\aleph_n} = \aleph_{n+2}$ for all $n < \omega$. Going further we show that we can obtain the failure of $\square^*_\kappa$ for all κ in the interval $[\aleph_1, \aleph_{\omega^2}]$ from suitable large cardinals, but at the cost that none of the $\aleph_{\omega \cdot m}$ are strong limit. Both of these results seem to require Lemma 0.1.

We make one notational remark before beginning the paper. We will write $V[\mathbb{P}]$ for a typical generic extension by a poset $\mathbb{P}$. Furthering this notation when we write $V[\mathbb{P}] \subseteq V[\mathbb{Q}]$ we mean that we can force over a given generic extension by $\mathbb{P}$ to obtain a generic extension by $\mathbb{Q}$.

1. Approximation lemmas

In this section we prove some preservation theorems. We start by defining the κ-approximation property.

Definition 1.1 *(Hamkins).* Let $V \subseteq W$ be models of set theory. The pair (V, W) has the *κ-approximation property* if and only if for every ordinal μ and every $b \subseteq \mu$ with $b \in W$, if for every $x \in \mathcal{P}_\kappa(\lambda)_V$, $b \cap x \in V$, then $b \in V$.

We say that a poset $\mathbb{P}$ has the κ-approximation property if and only if for every $\mathbb{P}$-generic G, $(V, V[G])$ has the κ-approximation property.

In [5] Hamkins proved:

Lemma 1.2. *Let β be a regular cardinal suppose that $|\mathbb{P}| = \beta$ and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is β^+-closed. If $\mathrm{cf}(\lambda) > \beta$, then $\mathbb{P} * \dot{\mathbb{Q}}$ does not add subsets to λ all of whose initial segments are in the ground model.*