Annals of Pure and Applied Logic 166 (2015) 1150–1164

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Equilibrium points of an AND–OR tree: Under constraints on probability

Toshio Suzuki *, Yoshinao Niida [1]

Department of Mathematics and Information Sciences, Tokyo Metropolitan University, Minami-Ohsawa, Hachioji, Tokyo 192-0397, Japan

ARTICLE INFO

Article history:
Received 10 July 2014
Received in revised form 8 June 2015
Accepted 2 July 2015
Available online 22 July 2015

MSC:
68T20
68Q17
03D15
68T27
68Q87
91A60

Keywords:
Game tree
AND–OR tree
OR–AND tree
Independent identical distribution
Computational complexity

ABSTRACT

We consider a depth-first search based algorithm to find the truth value of the root of an AND–OR tree. The cost is measured by the number of leaves probed during the computation. We consider probability distributions on the truth assignments, and restrict the attention to distributions that have the following properties: 1. The inputs come from a product distribution. 2. The output probability (of the root being assigned zero) is some fixed r such that $0 < r < 1$. Among such distributions we investigate the maximizer of the minimum expected cost. We show that the maximizer is an independent identical distributions (IID). The result may appear to be straightforward but, the fact is that the proof requires new ideas. As keys to the proof, we show fundamental relationships between the following two quantities: 1. The probability of the root being assigned zero. 2. The minimum expected cost. Our result justifies a claim in the paper of Liu and Tanaka (2007) [3].

1. Introduction

Given an algorithm for tree searching, the goal is to find the value of the root. The cost is measured by the number of leaves probed during the computation. The alpha–beta pruning algorithm is a well-known algorithm for tree searching. Knuth and Moore [2] pioneered the analysis of the alpha–beta pruning.

Baudet [1] and Pearl [4] studied the optimality of alpha–beta pruning in the case where values of leaves are independent and identically distributed. The optimality is established by Pearl [5] and Tarsi [8]. For more on important early works, see the references of [5].

* Corresponding author.
E-mail addresses: toshio-suzuki@tmu.ac.jp (T. Suzuki), yn.sputnik@gmail.com (Y. Niida).
[1] His current affiliation is Patent Result Co., Ltd., Japan.

http://dx.doi.org/10.1016/j.apal.2015.07.002

We are interested in the case of a uniform binary tree such that each leaf is bi-valued. Each min–max tree of this type can be viewed to be an AND–OR tree. Furthermore, the alpha–beta pruning is described in a simple way. In this case, the alpha–beta pruning algorithm is a depth-first search of the following property: Given an AND-node (an OR-node, respectively) v, if we know a child of v has value 0 (1, respectively) then we know that v has the same value, without probing the other child. Here, a cut-off (or, a skip) happens at the other child. We will review precise definitions of AND–OR trees, depth-first algorithms, and the alpha–beta pruning algorithm in the next section.

We are interested in the case where the associated probability distribution d on the truth assignments to the leaves is an independent distribution (ID) but d is not necessarily an independent identical distribution (IID). Here, an IID denotes an ID such that all the leaves have the same probability of having value 0.

Yao's principle [9], a variation of von-Neumann's minimax theorem, is useful for analyzing equilibriums of AND–OR trees. Saks and Wigderson [6] establish basic results on the equilibriums. Liu and Tanaka [3] have extended the works of Yao and Saks–Wigderson. They study distributions achieving the equilibrium. To be more precise, a distribution d_0 achieves the equilibrium if it satisfies the following. $\min_{A_D} C(A_D, d_0) = \max_d \min_{A_D} C(A_D, d)$. Here, C denotes the expected value of the cost. Recall that the cost is the number of leaves probed during the computation. A_D runs over all deterministic alpha–beta pruning algorithms. In the right-hand side, d runs over all distributions of certain type. In the case where d runs over of all IDs then d_0 is said to be the distribution achieving the equilibrium for IDs. We are going to give more precise definitions of equilibriums in the next section.

In the course of their study, Liu and Tanaka showed the following.

- Theorem 4 of Liu and Tanaka [3] (Theorem 6) If d achieves the equilibrium for IDs then d is an IID.

Liu and Tanaka write "it is not hard" to show the theorem, and omit the proof.

Convention on probability. Suppose that v is a node of a given tree, and that x is a real number between 0 and 1. Throughout the paper, unless otherwise specified, we write "*the probability of v is x*" or "*v has probability x*" to denote "v is assigned 0 with probability x".

In this paper, we show a stronger form of the above theorem.

- Main Theorem (Theorem 5) *Suppose that r is a real number such that $0 < r < 1$. Suppose that we restrict ourselves to distributions such that the probability of the root is r. Under this constraint, the statement of the above result of Liu and Tanaka still holds.*

This statement might appear at first glance to be a weaker form of Theorem 6. However it is actually a stronger statement. Informally speaking, a world champion (the distribution d in Theorem 6) is a local champion. By the phrase of "local champion", we denote the champion among distributions whose probabilities of the root are the same as that of d. In Section 5, we show that the probability for a world champion is neither 0 nor 1. Therefore, the result on an arbitrary local champion (Main Theorem) implies the result on an arbitrary world champion (Theorem 6).

Furthermore, in general, a result about the world champion does not imply a result about the local champion. To be more precise, given a world champion for height $h+1$, its restrictions to the child nodes of the root are not necessarily world champions for height h: This is immediately verified in the case of $h = 2$. Therefore, an induction by brute force on the height of trees does not work for a proof of Theorem 6.

Theorem 6 may appear to be not hard to prove on the surface but, as it can be noticed for the observation mentioned above, the fact is that the proof of Theorem 6 requires new ideas and is worth studying.

Our proof of Theorem 5 employs clever tricks of induction. In particular, we show the following two lemmas by induction.